# Mapping the Spacetime Metric with a Global Navigation Satellite 

## Final Report

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Date: February $28^{\text {th }}, 2010$

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Available on the ACT website http://www.esa.int/act

Ariadna ID: 09/1301
Study Type: Standard

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## InTRODUCTION

In this study we model a Global Navigation Satellite System (GNSS) in a Schwarzschild space-time, as a first approximation of the relativistic geometry around the Earth. Two very different ways of taking into account relativity in positioning systems are presented in the first chapter. In the second chapter closed time-like and scattering light-like geodesics are obtained analytically, describing respectively trajectories of satellites and electromagnetic signals. A method for ray-tracing in a weak gravitational field is presented. Then we implement an algorithm to calculate Schwarzschild coordinates of a GNSS user who receives proper times sent by four satellites, knowing their orbital parameters; the inverse procedure is implemented to check for consistency. The constellation of satellites therefore realizes a geocentric inertial reference system with no a priori realization of a terrestrial reference frame. We show that the calculation is very fast and could be implemented in a real GNSS, as an alternative to usual post-Newtonian corrections. Effects of non-gravitational perturbations on positioning errors are assessed, and methods to reduce them are sketched. In particular, inter-links between satellites could greatly enhance stability and accuracy of the positioning system.

Details of the calculations are usually put in the appendices for the comfort of the reader. The Mathematica algorithms are given in the appendices, and the codes written in Fortran are publicly available on the website atlas.estec.esa.int/ariadnet.

# GLOBAL NAVIGATION SATELLITE SYSTEMS AND RELATIVITY 

The classical concept of positioning systems for a Global Navigation Satellite System (GNSS) would work ideally if all satellites and the receiver were at rest in an inertial reference frame. But at the level of precision needed by a GNSS, one has to consider curvature and relativistic inertial effects of spacetime, which are far from being negligible.

In this chapter we will present two very different ways of including relativity in a positioning system: one way is to keep the newtonian conception of absolute time and space, and add a number of corrections depending on the desired accuracy; another way is to use a relativistic positioning system. This is a complete change of paradigm, as the constellation of satellites is described in a general relativistic framework. This new scheme for positioning could lead to numerous advantages: a very stable and accurate primary reference system, which could be used for many areas of science such as geology, gravitational wave detection or relativistic gravimetry.

### 1.1 The newtonian conception

The classical GPS In the newtonian theory space and time are absolute. Any pair of event has a causal relation: simultaneity has an absolute character. The space-time can be foliated with the hypersurfaces of simultaneous events (the space) by fixing the coordinate time, e.g. $t=t_{0}$. Then one can define an absolute time interval between two instants, and an absolute distance on the space. Let $c$ be the speed of an electromagnetic wave in the vacuum, and $\phi=-G M / r$ the gravitational potential of a central mass $M$, where $G$ is the gravitational constant and $r$ the radial distance to the central mass. Then this conception of space-time is a good approximation for an observer whose velocity $v \ll c$, and for a very weak gravitational field $\left|\phi / c^{2}\right| \ll 1$.

In the newtonian space a user needs ideally the signals of three satellites to locate himself. We assume that the clocks on board the three satellites $S_{i}(i=1,2,3)$ and the clock of the
user are perfect. Each satellite sends an electromagnetic wave to the user, where the time of emission $t_{i}$ of the signal is encoded. The user, having a clock, knows the time of reception $t_{R}$ of the signal. Then he can deduce its distance with respect to the three satellites: $c\left(t_{R}-t_{i}\right)$, which is a direct consequence of the finite speed of light. Therefore the user knows that he lies on a sphere of radius $c\left(t_{R}-t_{i}\right)$ centered on the satellite $S_{i}$. The three spheres centered on the three satellites intersect usually in two points, a problem easily solved with the method of trilateration. Then the position of the user is usually taken as the point being the closest to the surface of the Earth.

Let $(x, y, z)$ be the cartesian coordinates of the user in the Euclidean space $\mathbb{R}^{3}$, and $\left(x_{i}, y_{i}, z_{i}\right)$ the coordinates of the satellite $S_{i}$. We assume that the coordinates of the satellite $S_{i}$ are known at the time of emission $t_{i}{ }^{*}$. Then one has three unknowns and a system of three equations:

$$
\left(x-x_{i}\right)^{2}+\left(y-y_{i}\right)^{2}+\left(z-z_{i}\right)^{2}=c^{2}\left(t_{R}-t_{i}\right)^{2}, i=1,2,3,
$$

which can be solved by trilateration and gives 0,1 or 2 solutions.
The clocks on board the satellites are atomic cesium clocks. They have an error of 1.2 ns - for Galileo - after one day of operation (Waller et al. 2009). This leads potentially to an error of about 35 cm after one day. However, the bad clock of the user - compare to the one of the satellites - limits the precision of the positioning. It is then necessary to use other methods to correct for the user clock. These methods use four or more satellites.

Suppose that you know only poorly the time of reception of the signal. Then you add the time of reception $t$ as an unknown of the problem, and add one satellite to have a fourth equation. The unknown of the problem is the position of the user in the space-time ( $\mathrm{t}, \mathrm{x}, \mathrm{y}, \mathrm{z}$ ) and we have a system of four equations:

$$
\left(x-x_{\alpha}\right)^{2}+\left(y-y_{\alpha}\right)^{2}+\left(z-z_{\alpha}\right)^{2}=c^{2}\left(t-t_{\alpha}\right)^{2}, \alpha=1,2,3,4 .
$$

This system can be solved numerically with an iterative method that we will present in section 3.2, where only linear equations have to be solved.

Why include relativity? The theory of relativity (both special and general) teaches us that space and time are not absolute. A pair of events has a causal relation only if one is in the light cone of the other one. This has a lot of consequences on a GNNS, which is affected in three ways:

- in the equations of motion of the satellites;

[^0]- in the signal propagation;
- in the beat rate of the clocks.

For the GPS and the future Galileo, only the clocks effects are measurable, but they exceed largely the precision of the positioning system. The most important ones are:

- the gravitational frequency shift between the clocks (due to the local position invariance principle);
- the Doppler shift of the second order due to the motion of the satellites (special relativity).

The gravitational frequency shift implies that a clock runs faster when it is far away from a center of gravitational attraction. The Doppler shift of the second order implies that a clock in motion slows down. Then the clocks in the satellites will be slower than a clock on the ground (for an observer who is at rest compare to the ground clock). These two effects are opposite and have a net blue shift effect. These clocks effects imply an error of around 12 km after one day of operation, which is much more than the intended precision.

An order of magnitude The theory of general relativity is based upon the postulate of the Einstein Equivalence Principle, which can be be separated in three sub-principles (Will 2006): the Weak Equivalence Principle, the Local Lorentz Invariance and the Local Position Invariance (LPI). The LPI states that any local (non-gravitational) experiment is independent of where and when in the universe it is performed. From this principle one can infer that the proper time $\tau$ measured by a clock is given by (Will 1993)

$$
\begin{equation*}
c^{2} \mathrm{~d} \tau^{2}=\left(1-\frac{2 G M}{r c^{2}}\right) c^{2} \mathrm{~d} t^{2}-\mathrm{d} r^{2}-r^{2} \mathrm{~d} \varphi^{2} \tag{1.1}
\end{equation*}
$$

For a circular orbit we have $\mathrm{d} r=0$, so that relation (1.1) can be written

$$
\begin{equation*}
\left(\frac{\mathrm{d} \tau}{\mathrm{~d} t}\right)^{2}=\left(1-\frac{2 G M}{r c^{2}}\right)-\frac{v^{2}}{c^{2}} \tag{1.2}
\end{equation*}
$$

where we define $v=r \frac{\mathrm{~d} \varphi}{\mathrm{~d} t}$ the linear velocity. Let apply this formula for a clock in a satellite $S$ with proper time $\tau_{S}$, and for a fixed clock on the Earth with proper time $\tau_{R}$. Then the equation (1.2) implies

$$
\begin{equation*}
\left(\frac{\mathrm{d} \tau_{R}}{\mathrm{~d} \tau_{S}}\right)^{2}=\frac{1-\frac{2 G M}{r_{R} c^{2}}-\frac{v_{R}^{2}}{c^{2}}}{1-\frac{2 G M}{r_{S} c^{2}}-\frac{v_{S}^{2}}{c^{2}}} \tag{1.3}
\end{equation*}
$$

We suppose that we are in a weak gravitational field and the velocities are little compare to the speed of light. Then a first order approximation of the relation (1.3) leads to

$$
\frac{\mathrm{d} \tau_{R}}{\mathrm{~d} \tau_{S}}=1-\frac{G M}{r_{R} c^{2}}-\frac{v_{R}^{2}}{2 c^{2}}+\frac{G M}{r_{S} c^{2}}+\frac{v_{S}^{2}}{2 c^{2}}
$$

For the Galileo constellation, we obtain

$$
\begin{aligned}
-\frac{G M}{r_{R} c^{2}}+\frac{G M}{r_{S} c^{2}} & =-5.4554 \cdot 10^{-10} \\
-\frac{v_{R}}{2 c^{2}}+\frac{v_{S}}{2 c^{2}} & =+7.3715 \cdot 10^{-11}
\end{aligned}
$$

These numbers correspond to an error of about 12.2 km after one day of integration. The error for the GPS is a bit smaller, about 11.7 km . This is because the GPS satellites are on a slightly lower orbit than the Galileo ones, so the gravitational frequency shift is smaller.

A necessary change of paradigm Practically, the calculation is much more complicated (see Ashby (2003) or Pascual-Sánchez (2007) for an extended review). The "GPS coordinate time" $t_{\text {GPS }}$ is defined as the time of a clock at rest on the geoid. It has to be related to the time $t$ introduced in the equation (1.1), which can be interpreted as the time measured by a clock in an inertial frame at spatial infinity. Then one has to do transformations between the ECI (Earth Centered Inertial system) and the ECEF (Earth Centered Earth Fixed system). The orbital parameters of the satellite constellation are then expressed in the ECEF. To realize the ECEF a network of ground stations receiving the GNNS signals has been installed. The GPS uses the World Geodetic System 1984 (WGS-84); Galileo will use the Galileo Terrestrial Reference Frame (GTRF) (Altamimi 2009). These global reference frames are fixed to the Earth (via the ground stations) so their precision and stability in time are limited by our knowledge of the Earth dynamics. The main effects are plate tectonic motions, tidal effects on the Earth's crust and variations of the Earth rotation rate. In the WGS-84 the best accuracy achieved is 30 cm (NIMA 2000), with an average stability of $4 \mathrm{~cm} /$ year (Altamimi 2009). The use of other space geodetic techniques - VLBI, SLR and DORIS - is necessary to achieve a high precision ECEF. The International Terrestrial Reference Frame (ITRF), maintained by the International Earth Rotation and Reference Systems Service, combine efficiently these four techniques to reach a stability of about $1 \mathrm{~mm} /$ year, inferior by a factor of 10 to the science requirements (Altamimi 2009).

These considerations led Coll (Coll 2003) to propose the project "Système de Positionnement Relativiste" (SYPOR), an alternative to the scheme of usual positioning systems. The idea is to give to the constellation of satellites the possibility of constituting by itself a primary and autonomous positioning system, without any a priori realization of a ter-
restrial reference frame. This new positioning system leads to numerous advantages, among which we can cite:

- a better understanding of the principles of positioning systems;
- the new coordinates defined are measurable directly (they are observer independent). They constitute a physical coordinate system, which is not the case of the other coordinate system. This open new possibilities in experimental physics and astronomy;
- it can be used for extra-terrestrial navigation with the use of pulsars as clocks;
- with the use of satellites interlinks, the reference frame is very precise and stable: it could be used to detect gravitational waves and for high precision gravimetry and geology;
- it is a primary reference frame which is not tied to the Earth: it is independent of the Earth dynamics and continental drifts;
- the relativistic effects are already included in the definition of the positioning system, so there is no need to synchronize the clocks.


### 1.2 The relativistic positioning system

The null-coordinates To define a relativistic positioning system we have to introduce the "null-coordinates". They have been reintroduced recently by the works of Coll and Morales (1991), Rovelli (2002) and Blagojević et al. (2002). They have different names in the literature: "null-coordinates", "emission coordinates", "GPS coordinates", "GNSS coordinates". In this report we will use the first name which is a reference to their geometrical properties. The definition of these coordinates is rather simple, but they are a very powerful tool in general relativity. Let us define four particles $a=1,2,3,4$ coupled to general relativity. Their worldlines $\mathcal{C}_{a}$ are parametrized by their proper time $\tau^{a}$. We choose a random origin for $\tau=0$ on each world-line. Let $P$ be an arbitrary event. Then the past null cone of $P$ crosses each of the four worldlines in $\tau_{P}^{a}$ (see Fig.1.1). The quadruplet $\left(\tau^{1}, \tau^{2}, \tau^{3}, \tau^{4}\right)_{P}$ constitutes the null-coordinates of the event $P$.

The protocol to define null-coordinates can be seen in a different way. The worldine $\mathcal{C}_{a}$ of the particle $a$ defines a one-parameter family of future null cones, which can be parametrized by proper time $\tau_{A}$ (see Fig.1.2). The intersection of four future null cones $\tau^{a}$ from four worldines $\mathcal{C}_{a}$ defines an event with coordinates $\left(\tau^{1}, \tau^{2}, \tau^{3}, \tau^{4}\right)$. A user receiving these signals knows its position in this particular coordinate system.


Figure 1.1: Defining null-coordinates with the null past cone of a space-time event: let $P$ be an event in space-time. $\mathcal{C}_{a}$ is the worldline of a test particle $a$ parametrized by its proper time $\tau^{a}$; its origin $O$ is in $\tau^{a}=0$. The past null cone of the event $P$ cross $\mathcal{C}_{a}$ at the proper time $\tau_{P}^{a}$. With four different particles with the worldlines $\mathcal{C}_{a}(a=1,2,3,4)$, the past null cone of $P$ crosses the four worldlines in $\tau_{P}^{1}, \tau_{P}^{2}, \tau_{P}^{3}$ and $\tau_{P}^{4}$. Then $\left(\tau^{1}, \tau^{2}, \tau^{3}, \tau^{4}\right)_{P}$ are the null-coordinates of the event $P$.


Figure 1.2: Defining null-coordinates with four one-parameter families of null future cone: the worldline $\mathcal{C}_{a}$ defines a one-parameter family of null future cones. These null cones are hypersurfaces of $\tau^{a}=$ constant. The intersection of four null cones defines the event with coordinates $\left(\tau^{1}, \tau^{2}, \tau^{3}, \tau^{4}\right)$. The four particles can be chosen as four satellites of a constellation of satellites, broadcasting their proper time.

Properties of the null-coordinates The geometrical properties of the null-coordinates ( $\tau^{a}$ ) are such that (see Coll and Pozo (2006) for an detailed article)

- expressed in this coordinate system, the components of the contravariant metric tensor verifies $g^{a a}=0$, where $a=1,2,3,4$;
- i.e. the four families of coordinate hypersurfaces $\tau^{a}=$ constant are null hypersurfaces.

They are covariant and completely independent of any observer (Lachièze-Rey 2006). They define a primary reference system: there is no need to attach them to a Terrestrial reference system. If pulsars are used as clocks instead of the satellites clock, we have then a natural reference frame for deep space navigation. Relativity does not need to be added as corrections to these coordinates, as they are defined in a general relativistic framework. They are directly measurable, a real assets for practical uses such as localization, motion monitoring, geology, astrometry, cosmography and experimental gravitation.

These coordinates depend on the set of four satellites ones chooses and their dynamics. They are not "usual" coordinates, with one time and three space coordinates. However, they can be linked to a terrestrial reference system by the usual techniques discussed in the previous section. The conceptual difficulty does not lie anymore in the conception of the primary reference frame but in its link with terrestrial reference frames. This allows to control much more precisely all the perturbations that limit the accuracy and the stability of the primary reference frame. Indeed, it is sufficient to know the effect of (non-gravitational and gravitational) perturbations on the dynamics of the satellites in order to characterize this primary reference frame.

Review of recent literature Coll and collaborators (Coll et al. 2006a,b) studied relativistic positioning systems in the case of a two-dimensional space-time for geodesic emitters in a Minkowski plane and for static emitters in the Schwarzschild plane. A relativistic positioning system has been studied in the vicinity of the Earth: calculations were performed to first order in a Schwarzschild spacetime (Bahder 2001; Ruggiero and Tartaglia 2008). A "galactic reference system" has been studied, where timing signals received by four pulsars were considered as null-coordinates (Coll and Tarantola 2003; Tartaglia et al. 2010). The next generation of GNSS will have cross-link capabilities (Directorate of the Galileo Programme and Navigation Related Activities 2009). Each satellite will broadcast proper time to other satellites in view, as well as their proper time. With this information, one could in principle map the spacetime geometry in the vicinity of the constellation of satellites by solving an inverse problem (Tarantola et al. 2009).

## SCHWARZSCHILD SPACE-TIME

The Schwarzschild space-time is a good approximation of the geometry around the Earth. The local inertial coordinate system is tied to the center of the Earth and oriented into 4 mutually orthogonal directions $t, X, Y$, and $Z .{ }^{*}$ The Schwarzschild space-time is usually represented by the metric in spherical coordinates $t, r, \theta$, and $\varphi$, such that $X=r \sin \theta \cos \phi$, $Y=r \sin \theta \sin \phi$, and $Z=r \cos \theta$. In these coordinates the metric is

$$
g_{\mu \nu}=\left[\begin{array}{cccc}
-\left(1-\frac{2 M}{r}\right) & 0 & 0 & 0  \tag{2.1}\\
0 & \frac{1}{1-2 M / r} & 0 & 0 \\
0 & 0 & r^{2} & 0 \\
0 & 0 & 0 & r^{2} \sin ^{2} \theta
\end{array}\right]
$$

We use natural units ${ }^{\dagger} c=G=1$. Geodesics are governed by the Hamiltonian:

$$
\begin{equation*}
H=\frac{1}{2}\left[-\frac{1}{1-\frac{2 M}{r}} p_{t}^{2}+\left(1-\frac{2 M}{r}\right) p_{r}^{2}+\frac{1}{r^{2}}\left(p_{\theta}^{2}+\frac{1}{\sin ^{2} \theta} p_{\phi}^{2}\right)\right] \tag{2.2}
\end{equation*}
$$

which admits 8 constants of motion: value of Hamiltonian $(H)$ and Lagrangian $(L)$, controlling the relation between time and distance, energy $E=p_{t}$, the three components of angular momentum $(\vec{l})$, longitude of periapsis $(\omega)$ and time of periapsis passage $\left(t_{p}\right)$. It is convenient to introduce another local inertial (right-handed) orthonormal tetrad $\hat{n}, \hat{e}_{1}$ and $\hat{e}_{2}$ (c.f. Fig. 2.1), where $\hat{n}$ is the constant unit vector pointing in the direction of the angular momentum: $\vec{l}=l \hat{n}$. The two unit vectors $\hat{e}_{1}$ and $\hat{e}_{2}$ in the orbital plane are such that $\hat{e}_{1}$ points in the direction of the initial perigee. The components of these vectors with respect

[^1]

Figure 2.1: The orbital plane in equatorial coordinates: $\hat{n}$ unit normal, $\iota$ inclination, $\Omega$ longitude of the ascending node, $\omega$ longitude of periapsis and $\lambda$ true anomaly.
to the local Cartesian coordinate basis are expressed as:

$$
\begin{align*}
\hat{e}_{1} & =(\cos \omega \cos \Omega-\cos \iota \sin \omega \sin \Omega, \quad \cos \omega \sin \Omega+\cos \iota \sin \omega \cos \Omega, \sin \iota \sin \omega) \\
\hat{e}_{2} & =(\sin \omega \cos \Omega-\cos \iota \cos \omega \sin \Omega,-\sin \omega \sin \Omega+\cos \iota \cos \omega \cos \Omega, \sin \iota \cos \omega) \\
\hat{n} & =(\sin \iota \sin \Omega,-\sin \iota \cos \Omega, \cos \iota) \tag{2.3}
\end{align*}
$$

where $\Omega$ is the longitude of the ascending node and $\iota$ is the inclination of the orbit with respect to the $X-Y$ plane.

The only parameter changing along the orbit is the true anomaly $(\lambda)$, which obeys the differential orbit equation:

$$
\begin{equation*}
\frac{\mathrm{d} u}{\mathrm{~d} \lambda}= \pm \sqrt{A^{2}-u^{2}(1-u)+B(1-u)} \tag{2.4}
\end{equation*}
$$

where $u=2 M / r$, and $A=2 M E / l$ and $B=2 H(2 M / l)^{2}$ are two constants of motion related to orbital energy and orbital angular momentum. After (2.4) is solved for $u$ as a function of $\lambda$, the orbit can be described with

$$
\begin{equation*}
\vec{r}(\lambda)=\frac{2 M}{u}\left(\hat{e}_{1} \cos \lambda+\hat{e}_{2} \sin \lambda\right) . \tag{2.5}
\end{equation*}
$$

The spherical coordinates $\theta$ and $\phi$ along the orbit are expressed as* (Čadež and Gomboc 1996, Eqs. 6 - 16):

$$
\begin{align*}
\cos \theta & =\sin \iota \sin (\lambda+\omega)  \tag{2.6}\\
\tan \frac{\varphi-\Omega}{2} & =\frac{\cos \iota \sin (\lambda+\omega)}{\sin \theta+\cos (\lambda+\omega)} \tag{2.7}
\end{align*}
$$

Time and proper time obey the following differential equations:

$$
\begin{align*}
\frac{\mathrm{d} t}{\mathrm{~d} u} & =\frac{2 M A}{u^{2}(1-u) \sqrt{A^{2}-u^{2}(1-u)+B(1-u)}}  \tag{2.8a}\\
\frac{\mathrm{d} \tau}{\mathrm{~d} u} & =\frac{2 M A}{E} \frac{1}{u^{2} \sqrt{A^{2}-u^{2}(1-u)+B(1-u)}} \tag{2.8b}
\end{align*}
$$

The differential equations (2.4)-(2.8a) are formally the same for light-like (where $B=0$ ) and time-like orbits. However, solutions depend on the type of orbit, e.g. closed, scattering or plunging (details in App. A). In the following sections we give solutions for closed timelike orbits and scattering light-like orbits, which can be used to model satellites and photons trajectory in a GNSS.

### 2.1 Time-like geodesics

GNSS satellites are on closed time-like orbits. After solving (2.4) for such case, we obtain the orbit equation

$$
\begin{equation*}
u(\lambda)=U_{2}-\left(U_{2}-U_{3}\right) \mathrm{cn}^{2}\left(\left.\mathrm{~K}\left(m_{a}\right)+\frac{\lambda}{2 n_{a}} \right\rvert\, m_{a}\right) \tag{2.9}
\end{equation*}
$$

where $U_{1}, U_{2}, U_{3}$ are the roots of the polynomial $P(u)=A^{2}-u^{2}(1-u)+B(1-u)$, and $n_{a}$ and $m_{a}$ are functions of them. K and cn are the complete elliptic integral of the first kind and Jacobian elliptic function respectively. The true anomaly $\lambda$ is defined as in the Keplerian case; at periapsis it has values $\lambda_{p}=4 n_{a} \mathrm{~K}\left(m_{a}\right) k$, where $k$ is an integer; $U_{2}$ and $U_{3}$ are related to the radii of the periapsis $r_{\mathrm{p}}=2 M / U_{2}$ and apoapsis $r_{\mathrm{a}}=2 M / U_{3}$ respectively. Obviously, for circular orbits $U_{2}=U_{3}$, and the orbit equation reduces to $r=$ const. $=r_{\mathrm{a}}=r_{\mathrm{p}}$. The numerical procedure to calculate space-time position on the orbit is given in the App. A. 1 and App. B on page 46.

Schwarzschild time (A.12) and proper time (A.13) along the orbit are obtained by integrating equations (2.8a) - (2.8b). The solutions of time-like geodesic equation are illustrated

[^2]

Figure 2.2: Left: A time-like orbit with semi-major axis $a=500 r_{\mathrm{g}}$ and eccentricity $\varepsilon=0.3$. Middle: Time (black) and proper time (red) for the same orbit. Right: The difference between time and proper time for the same orbit. All values of $r, t$, and $\tau$ are in units of $M$.
in Fig.2.2. Orbital parameters were intentionally chosen such that relativistic effects are clearly visible, i.e. periapsis precession and the difference between time and proper time.

For the values of orbital parameters of the Galileo satellites, the difference between coordinate time and proper time goes up to $10 \mu s$ per orbit.

### 2.2 Light-like geodesics

In the eikonal approximation, electromagnetic signals sent by satellites follow null scattering geodesics. After solving (2.4) with $B=0$ for such cases, we obtain the orbit equation

$$
\begin{equation*}
u(\lambda)=u_{2}-\left(u_{2}-u_{3}\right) \mathrm{cn}^{2}\left(\left.\mathrm{~K}(m)+\frac{\lambda}{n} \right\rvert\, m\right), \tag{2.10}
\end{equation*}
$$

where the true anomaly $\lambda$ takes values on the interval $\lambda \in\left(\mathrm{F}\left(\chi_{\max } \mid m\right)-\mathrm{K}(m), \mathrm{F}\left(\chi_{\min } \mid m\right)-\right.$ $\mathrm{K}(m))$. Here $\chi_{\text {min }}=\arccos \left(\sqrt{u_{2} /\left(u_{2}-u_{3}\right)}\right), \chi_{\max }=\arccos \left(-\sqrt{u_{2} /\left(u_{2}-u_{3}\right)}\right), \mathrm{F}$ is the elliptic integral of the first kind, and $u_{1}, u_{2}, u_{3}$ are the roots of the polynomial $P(u)=$ $A^{2}-u^{2}(1-u)$. The constants $u_{1}, u_{2}, u_{3}, m, n$ depend only on one constant of motion $A=2 M E / l$, which is the inverse of the impact parameter (see App. A.2).

Coordinate time (A.19) as a function of $\lambda$ is obtained by integrating Eq.(2.8a) with $B=0$. In the case of GNSS the gravitational field is very weak, so photon orbits are essentially straight lines. Differences between the relativistic and non-relativistic time-offlight are of the order of 1 ns when considering a signal traveling from a satellite to the user.

In order to define null-coordinates, one needs to find the time-of-flight of a photon connecting two given events $\mathcal{P}_{i}=\left(t_{i}, x_{i}, y_{i}, z_{i}\right)$ and $\mathcal{P}_{f}=\left(t_{f}, x_{f}, y_{f}, z_{f}\right)$. To do this, we follow Čadež and Kostić (2005, hereafter ČK05).


Figure 2.3: Testing the time-of-flight routine in highly curved space-time. The calculated orbit is shown in red.

### 2.3 Ray-tracing

The constants of motion describing light-like geodesics are $A=2 M E / l, \hat{n}$, and $\omega$, where $l$ and $E$ are orbital angular momentum and energy respectively, $\hat{n}$ is the direction of the angular momentum, and $\omega$ the longitude of the periapsis. Since orbits are planar, $\hat{n}$ of an orbit going through $\mathcal{P}_{i}$ and $\mathcal{P}_{f}$ is automatically calculable from these data. Undetermined remain only the constants $A$ and $\omega$. Writing the orbit equation at $\mathcal{P}_{i}$ and $\mathcal{P}_{f}$, we obtain two non-linear equations for these two non-trivial constants of motion. However, since the longitude of the periapsis occurs only linearly as the argument of elliptic functions, it is possible to use elliptic functions addition theorem to eliminate the longitude of the periapsis and obtain a single non-linear equation for the orbital parameter $A$, as a function of initial and final coordinates, i.e. $r_{i}, r_{f}$, and $\Delta \lambda$, where $r_{i}$ and $r_{f}$ are the distance of the emitter and the receiver from the centre of the Earth, and $\Delta \lambda$ is the angle between the vectors corresponding to $\vec{r}_{i}$ and $\vec{r}_{f}$.

This method is described in App. A. 3 and the numerical procedure on page 50. Once the value of this parameter is known, it is straightforward to get the time-of-flight $\Delta t$ between the initial and the final point

$$
\begin{equation*}
\Delta t=\left.t\right|_{\lambda=\lambda_{f}}-\left.t\right|_{\lambda=\lambda_{i}}, \tag{2.11}
\end{equation*}
$$

where the times $t$ at points with $\lambda=\lambda_{f}$ and $\lambda=\lambda_{i}$ on the light-like geodesic with previously determined $A$ are calculated as described in App. A.3.

An example of orbit determination from $\Delta \lambda, r_{i}$, and $r_{f}$ is shown in Fig. 2.3.
In the case of the Galileo constellation, the delay due to bending of the light ray goes up to 1 ns .

## Numerical algorithms

We assume that the positions of all satellites as functions of time and their proper time are exactly calculable, i.e. given the proper time $\tau_{i}$ of the $i$-th satellite, we can exactly (to any precision required) calculate the four space-time coordinates ( $t_{i}, x_{i}, y_{i}, z_{i}$ ) of this satellite. The coordinates $\left(t_{i}, x_{i}, y_{i}, z_{i}\right)$ are referred to the common Schwarzschild coordinate system centred at the centre of the Earth. An observer who wants to determine his position in spacetime ( $t_{o}, x_{o}, y_{o}, z_{o}$ ) receives proper time signals from four (or more) satellites that constantly broadcast the time from their proper time clocks. At the moment of reception, the four time signals received are $\left(\tau_{1}, \tau_{2}, \tau_{3}, \tau_{4}\right)$ (see section 1.2).

During this study, we developed two algorithms:

1. An algorithm that calculates null-coordinates $\left(\tau_{1}, \tau_{2}, \tau_{3}, \tau_{4}\right)$ from space-time coordinates ( $t_{o}, x_{o}, y_{o}, z_{o}$ ) of an observer (or a receiving satellite).
2. And the "reverse" algorithm that calculates space-time coordinates ( $t_{o}, x_{o}, y_{o}, z_{o}$ ) of an observer (or a receiving satellite) from its null-coordinates $\left(\tau_{1}, \tau_{2}, \tau_{3}, \tau_{4}\right)$.

The combination of the two algorithms can be used to test their accuracy in the following way: if the first algorithm is used to calculate null-coordinates and then the second one to calculate Schwarzschild coordinates from these null-coordinates, i.e. $\left(t_{o}, x_{o}, y_{o}, z_{o}\right) \rightarrow$ $\left(\tau_{1}, \tau_{2}, \tau_{3}, \tau_{4}\right) \rightarrow\left(t_{o}^{\prime}, x_{o}^{\prime}, y_{o}^{\prime}, z_{o}^{\prime}\right)$, the resulting Schwarzschild coordinates should be the same as the ones that were used in the first algorithm, i.e. $\left(t_{o}^{\prime}, x_{o}^{\prime}, y_{o}^{\prime}, z_{o}^{\prime}\right)=\left(t_{o}, x_{o}, y_{o}, z_{o}\right)$.

### 3.1 Calculating null-coordinates from Schwarzschild coordinates

If the constants of motion for all satellites are known, it is possible to calculate their positions and times for any value of the true anomaly $\lambda$. A user at point $\mathcal{P}_{o}=\left(t_{o}, x_{o}, y_{o}, z_{o}\right)$ receives the signals from four satellites, which sent their signals at points $\mathcal{P}_{i}=\left(t_{i}, x_{i}, y_{i}, z_{i}\right)$ determined


Figure 3.1: Determining null-coordinates from Schwarzschild coordinates. Point of receiving the signal is marked with $P_{o}$. The point of emission $P_{i}$ has to be determined by connecting the two points with a light-like geodesic (diagonal line). For clarity, only one spatial coordinate is shown.
by $\lambda_{i}$. The null coordinates of the user at $\mathcal{P}_{o}$ are the proper times $\tau_{i}(i=1, \ldots, 4)$ of the sending satellites at $\mathcal{P}_{i}$ (see Fig. 3.1).

Since the proper time $\tau_{i}$ depends on the true anomaly $\lambda_{i}$, we calculate $\lambda_{i}$ at the emission point $\mathcal{P}_{i}$ from the equation that connects $\mathcal{P}_{o}$ and $\mathcal{P}_{i}$ with a light-like geodesic

$$
\begin{equation*}
t_{o}-t_{i}\left(\lambda_{i}\right)=T_{f}\left(\vec{R}_{i}\left(\lambda_{i}\right), \vec{R}_{o}\right), \tag{3.1}
\end{equation*}
$$

where $\vec{R}_{i}=\left(x_{i}, y_{i}, z_{i}\right)$ and $\vec{R}_{o}=\left(x_{o}, y_{o}, z_{o}\right)$ are respectively the spatial vectors of the sending satellite and the receiving user. The function $T_{f}$ calculates the time-of-flight of photons between $\mathcal{P}_{o}$ and $\mathcal{P}_{i}$ as described in full detail in Sec. 2.3 and App. A.3. Since $t_{i}$ and $\vec{R}_{i}$ are functions of $\lambda_{i}$ only, the above equation is an equation for $\lambda_{i}$ and can be very efficiently solved by known numerical algorithms, e.g. Newton method. Once the value of $\lambda_{i}$ is determined, it is straightforward to calculate proper time of emission $\tau_{i}$ from (A.13) and (A.14) for each sending satellite and thus obtain the four null-coordinates of the user at $\mathcal{P}_{o}=\left(\tau_{1}, \tau_{2}, \tau_{3}, \tau_{4}\right)$.

### 3.2 Calculating Schwarzschild coordinates from nullcoordinates

Here we solve the inverse problem: calculate Schwarzschild coordinates of the event $\mathcal{P}_{o}$ from $\left(\tau_{1}, \tau_{2}, \tau_{3}, \tau_{4}\right)$ sent by the four satellites. As we assume that the constants of motion of all satellites are known, we can deduce their space-time positions $\mathcal{P}_{i}=\left(t_{i}, x_{i}, y_{i}, z_{i}\right)$ from proper times $\tau_{i}$. Events $\mathcal{P}_{i}=\left(t_{i}, x_{i}, y_{i}, z_{i}\right)$ and $\mathcal{P}_{o}=\left(t_{o}, x_{o}, y_{o}, z_{o}\right)$ are connected with light-like


Figure 3.2: The four green points represent the four satellites at $\vec{R}_{i}$ and the red point is the user.
geodesics. In a flat space-time, they solve the four equations

$$
\begin{equation*}
t_{o}-t_{i}=\sqrt{\left(x_{i}-x_{o}\right)^{2}+\left(y_{i}-y_{o}\right)^{2}+\left(x_{i}-x_{o}\right)^{2}} \tag{3.2}
\end{equation*}
$$

These four equations can be solved for $\left(t_{o}, x_{o}, y_{o}, z_{o}\right)$ by a geometric construction. Let $\vec{R}_{i}=\left(X_{i}, Y_{i}, Z_{i}\right)$ be the spatial coordinates vectors of the satellites at $\mathcal{P}_{i}$. The situation is illustrated in Fig. 3.2, where the four green points represent the four satellites at $\vec{R}_{i}$ and the red point is the user.

The 4 spheres centred at $\vec{R}_{i}$ have radii $\left(t_{o}-t_{i}\right)$. Thus, the user is at the intersection of the four spheres. To find his position, we proceed as follows: suppose that the observer's dead reckoning coordinates are $\left(t_{o}^{(0)}, x_{o}^{(0)}, y_{o}^{(0)}, z_{o}^{(0)}\right)$. The radii of the four spheres centred at $\vec{R}_{i}$ would then be $\left(t_{o}^{(0)}-t_{i}\right)$. In general, these four spheres have no common point. However, the dead reckoning position can always be chosen in such a way that any two spheres intersect. Consider the planes defined by the circle of intersection of sphere 1 and 2 and sphere 3 and 4. These two planes generally intersect along a straight line; call it line 1 . Next consider the intersection of spheres 1 and 3 , and spheres 2 and 4 . The corresponding planes intersect along a straight line called line 2. If (3.2) is satisfied, then line 1 and line 2 intersect at the position of the user. However, since $\left(t_{o}^{(0)}, x_{o}^{(0)}, y_{o}^{(0)}, z_{o}^{(0)}\right)$ is not yet the solution of (3.2), the
lines 1 and 2 generally bypass each other. We calculate the positions on both lines where they meet at closest distance. The geometrical centre of the two positions is taken as the better approximation for the spatial position of the observer $\vec{R}_{o}^{(1)}$, and the distance between the two points of closest passage $(\mathrm{d} \vec{P})$ is a measure of how close we are to the solution. In the next step, we repeat the procedure with a slightly different $t_{o}$ to find the derivative of $\mathrm{d} \vec{P}$ with respect to $t_{o}$. This derivative is then used in Newton's method to find a new $t_{o}^{(1)}$ with a smaller $\mathrm{d} \vec{P}$. With this value a new spatial position $\vec{R}_{o}^{(2)}$ is calculated and the procedure is repeated until $\mathrm{d} \vec{P}$ becomes less than the value prescribed by the accuracy requirement. After $n$ steps $(n \sim 5)$, the procedure converges to $\left\{t_{o}^{(n)}, \vec{R}_{o}^{(n)}\right\}$, the solution of (3.2). Note, however, that a unique solution exists only if the normals to all planes whose intersections define lines 1 and 2 do not lie in a plane, i.e. if the four satellites do not lie in one plane.

However, eq. (3.2) does not take into account the gravitational time delay and the bending of light, so our result is not exact. For realistic positions of Galileo satellites the position error is a few $10^{-9}$ orbital radii, i.e. a few centimetres. The final correction to the position is made by solving the relativistic equations of light propagation from the satellites to the user in the following form:

$$
\begin{equation*}
t_{o}-t_{i}=T_{f}\left(\vec{R}_{i}, \vec{R}_{o}\right) \tag{3.3}
\end{equation*}
$$

where $T_{f}\left(\vec{R}_{i}, \vec{R}_{o}\right)$ calculates the coordinate time for the light to travel from $\vec{R}_{i}$ to $\vec{R}_{o}$. Taking $\left\{t_{o}^{(n)}, \vec{R}_{o}^{(n)}\right\}$ as an initial approximation, we use a generalization of the classical stellar navigation solution to solve (3.3). This equation is written in the form:

$$
\begin{align*}
t_{o}^{(n)}+\mathrm{d} t-t_{i} & =T_{f}\left(\vec{R}_{i}, \vec{R}_{o}^{(n)}+\mathrm{d} \vec{R}\right) \\
& =T_{f}\left(\vec{R}_{i}, \vec{R}_{o}^{(n)}\right)+\vec{\nabla}_{o} T_{f}\left(\vec{R}_{i}, \vec{R}_{o}^{(n)}\right) \cdot \mathrm{d} \vec{R}+\mathcal{O}\left(\mathrm{d} \vec{R}^{2}\right) \tag{3.4}
\end{align*}
$$

where $\vec{\nabla}_{o}$ is the gradient with respect to the position of the user $\vec{R}_{o}$. We now assume that the gravitational field is weak, i.e. $|M / R| \ll 1$. Then

$$
T_{f}\left(\vec{R}_{i}, \vec{R}_{o}\right)=\left|\vec{R}_{o}-\vec{R}_{i}\right|+\mathcal{O}(M)
$$

and

$$
\begin{align*}
\vec{\nabla}_{o} T_{f}\left(\vec{R}_{i}, \vec{R}_{o}\right) & =\frac{\vec{R}_{o}-\vec{R}_{i}}{\left|\vec{R}_{o}-\vec{R}_{i}\right|}+\mathcal{O}(M / R) \\
& =\hat{u}_{i}+\mathcal{O}(M / R) \tag{3.5}
\end{align*}
$$

where $\hat{u}_{i}$ is the unit vector pointing from satellite $i$ to the supposed position of the user.


Figure 3.3: The orbits of 4 satellites around the Earth. The satellites at initial positions are marked with black dots, and the observer on Earth's surface with a red dot. Green sphere represents the Earth. The sizes of satellites are not to scale.

| satellite | $\Omega\left[^{\circ}\right]$ | $\omega\left[^{\circ}\right]$ | $\iota\left[^{\circ}\right]$ | $a\left[\mathrm{r}_{\mathrm{g}}\right]$ | $\varepsilon$ | $t_{p}[\mathrm{rg} / \mathrm{c}]$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 0 | 0 | 45 | $5 \times 10^{9}$ | $1.1 \times 10^{-9}$ | 0 |
| 2 | 30 | 5 | 45 | $5 \times 10^{9}$ | $1.1 \times 10^{-9}$ | 0 |
| 3 | 60 | 10 | 45 | $5 \times 10^{9}$ | $1.1 \times 10^{-9}$ | 0 |
| 4 | 90 | 15 | 45 | $5 \times 10^{9}$ | $1.1 \times 10^{-9}$ | 0 |

Table 3.1: Orbital parameters for 4 satellites: longitude of ascending node $(\Omega)$, longitude of perigee $(\omega)$, inclination $(\iota)$, major semi-axis $(a)$, eccentricity $(\varepsilon)$, and time of perigee passage $\left(t_{p}\right)$.

Eq. (3.3) now becomes a set of four linear equations for $\mathrm{d} t$ and the three components of $\mathrm{d} \vec{R}$ :

$$
\begin{equation*}
t_{o}^{(n)}-t_{i}-T_{f}\left(\vec{R}_{i}, \vec{R}_{o}^{(n)}\right)=-\mathrm{d} t+\vec{u}_{i}^{(n)} \cdot \mathrm{d} \vec{R} \tag{3.6}
\end{equation*}
$$

The error of the corrected position decreases by 6 to 9 orders of magnitude - bringing the Galileo position error to the order of micrometers. If necessary, this error can be decreased even further by solving (3.6) again with $t_{o}^{(n+1)}=t_{o}^{(n)}+\mathrm{d} t$ and

$$
\begin{equation*}
\vec{R}_{o}^{(n+1)}=\vec{R}_{o}^{(n)}+\mathrm{d} \vec{R} \tag{3.7}
\end{equation*}
$$

The implementation of this algorithm in the language of Mathematica is shown in App. B.

### 3.3 Accuracy and speed of the algorithms

The above algorithms were tested in a simulation of 4 satellites $S_{i}(i=1, \ldots, 4)$ in orbit around the Earth communicating with a static user on the Earth's surface*. The only input parameters are the orbital parameters of the four satellites (table 3.1) and the coordinates of the user. The orbits for the four satellites are illustrated in Fig. 3.3. The Schwarzschild

[^3]coordinates of the user are
$$
r_{o}=1.595 \cdot 10^{9} M, \theta_{o}=43.97^{\circ}, \phi_{o}=14.5^{\circ}
$$

The simulation runs in the following way. At every time-step

$$
t^{(n)}=t^{(n-1)}+\Delta t, \quad n=2,3 \ldots
$$

where $\Delta t=6 \cdot 10^{12} M$ and $n$ counts the time-steps of the simulation:

1. We calculate null coordinates $\left(\tau_{1}^{(n)}, \tau_{2}^{(n)}, \tau_{3}^{(n)}, \tau_{4}^{(n)}\right)$ of the user from his Schwarzschild coordinates $\left(t^{(n)}, x_{o}^{(n-1)}, y_{o}^{(n-1)}, z_{o}^{(n-1)}\right)$, as described in Sec. 3.1. (For $n=1$ we choose $\left.\left(t^{(1)}, x_{o}^{(0)}, y_{o}^{(0)}, z_{o}^{(0)}\right)=\left(0, r_{o}, \theta_{o}, \phi_{o}\right)\right)$.
2. From previously calculated null coordinates $\left(\tau_{1}^{(n)}, \tau_{2}^{(n)}, \tau_{3}^{(n)}, \tau_{4}^{(n)}\right.$ ), we calculate Schwarzschild coordinates $\left(t_{i}^{(n)}, x_{i}^{(n)}, y_{i}^{(n)}, z_{i}^{(n)}\right.$ ) for every satellite $S_{i}$ at its new position $\tau_{i}^{(n)}$ by numerically solving the equation $\tau\left(\lambda_{i}^{(n)}\right)=\tau_{i}^{(n)}$ for $\lambda_{i}^{(n)}$.
3. From these Schwarzschild coordinates $\left(t_{i}^{(n)}, x_{i}^{(n)}, y_{i}^{(n)}, z_{i}^{(n)}\right)$, we calculate Schwarzschild coordinates $\left(t_{o}^{(n)}, x_{o}^{(n)}, y_{o}^{(n)}, z_{o}^{(n)}\right)$ of the user as described in Sec. 3.2.

The numerical code of this simulation is written in Fortran 90 and is publicly available on the website atlas.estec.esa.int/ariadnet. This code can be easily generalized to include a moving user, more satellites or communications between all the satellites.

## Accuracy

The accuracy of these algorithms is tested by comparing the initial Schwarzschild coordinates of the user with his Schwarzschild coordinates calculated at each time step. As the user is static, his coordinates $\left(x_{o}, y_{o}, z_{o}\right)$ should be constant during the simulation.

In table 3.2 we show the relative errors of all the coordinates, defined as

$$
\begin{equation*}
\epsilon_{t}^{(n)}=\frac{t^{(n)}-t_{o}^{(n)}}{t^{(n)}} \tag{3.8}
\end{equation*}
$$

and

$$
\begin{equation*}
\epsilon_{x}^{(n)}=\frac{x_{o}^{(1)}-t_{o}^{(n)}}{x_{o}^{(1)}}, \epsilon_{y}^{(n)}=\frac{y_{o}^{(1)}-t_{o}^{(n)}}{y_{o}^{(1)}}, \epsilon_{z}^{(n)}=\frac{z_{o}^{(1)}-t_{o}^{(n)}}{z_{o}^{(1)}} . \tag{3.9}
\end{equation*}
$$

The relative error of the coordinate $t$ is $\sim 10^{-32}$, and relative errors of the coordinates $x, y$, and $z$ are few orders of magnitude larger, i.e. $\sim 10^{-25}-10^{-27}$ (see table 3.2). In this table we also show the determinant det of the matrix, which is used for calculating the space-time coordinates.

| $n$ | $\epsilon_{t}$ | $\epsilon_{x}$ | $\epsilon_{y}$ | $\epsilon_{z}$ | $d e t$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 430 | $1.93758 \mathrm{E}-32$ | $-6.15473 \mathrm{E}-27$ | $-7.34498 \mathrm{E}-26$ | $-4.86318 \mathrm{E}-26$ | $1.72246 \mathrm{E}-003$ |
| 431 | $-2.99206 \mathrm{E}-32$ | $-1.78472 \mathrm{E}-26$ | $-7.64415 \mathrm{E}-26$ | $1.84489 \mathrm{E}-25$ | $1.59076 \mathrm{E}-003$ |
| 432 | $-1.46740 \mathrm{E}-31$ | $-4.82243 \mathrm{E}-26$ | $-1.13881 \mathrm{E}-25$ | $7.31360 \mathrm{E}-25$ | $1.46187 \mathrm{E}-003$ |
| 433 | $-1.72335 \mathrm{E}-32$ | $-1.09471 \mathrm{E}-26$ | $-7.03668 \mathrm{E}-26$ | $1.27280 \mathrm{E}-25$ | $1.33591 \mathrm{E}-003$ |
| 434 | $-1.01827 \mathrm{E}-32$ | $-7.65914 \mathrm{E}-27$ | $-5.69550 \mathrm{E}-26$ | $9.27816 \mathrm{E}-26$ | $1.21302 \mathrm{E}-003$ |

Table 3.2: Relative errors of the coordinates as defined in (3.8) - (3.9), for the last five timesteps of the simulation, which is equivalent to more than one orbit of the satellites. The determinant det of of matrix used for calculating space-time position is also shown in the last column.

The algorithm for determining Schwarzschild coordinates from null coordinates (Sec. 3.2) works only if the four satellites are not in the same plane, i.e. the determinant det must be non-zero. The opposite may happen during a simulation in which case the resulting position has a very large error or is left undetermined. As an example, we show in table 3.3 relative errors where the positions cannot be determined for a given configuration of the satellites, which is reflected in the close-to-zero negative values of det, as well as in a jump in the relative errors from $\sim 10^{-31}$ to $\sim 10^{-2}$ for the coordinate $t$, and from $\sim 10^{-25}$ to $\sim 10^{+3}$ for the coordinate $z$. In the case of GNSS satellites such a situation should never occur, since there are always more than four satellites visible and it should always be possible to choose four that do not lie in the same plane.

| $n$ | $\epsilon_{t}$ | $\epsilon_{x}$ | $\epsilon_{y}$ | $\epsilon_{z}$ | $d e t$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 7 | $-1.11018 \mathrm{E}-30$ | $3.21695 \mathrm{E}-27$ | $-8.46151 \mathrm{E}-26$ | $1.88565 \mathrm{E}-25$ | $5.81128 \mathrm{E}-003$ |
| 8 | $-9.83688 \mathrm{E}-31$ | $3.86661 \mathrm{E}-27$ | $-9.02631 \mathrm{E}-26$ | $2.02826 \mathrm{E}-25$ | $5.85868 \mathrm{E}-003$ |
| 9 | $-9.29760 \mathrm{E}-31$ | $3.02395 \mathrm{E}-27$ | $-8.94093 \mathrm{E}-26$ | $2.02005 \mathrm{E}-25$ | $5.90283 \mathrm{E}-003$ |
| 10 | $-5.15060 \mathrm{E}-02$ | $-7.85972 \mathrm{E}+02$ | $-7.40892 \mathrm{E}+01$ | $-2.31309 \mathrm{E}+03$ | $-3.37037 \mathrm{E}-014$ |
| 11 | $-1.51029 \mathrm{E}-02$ | $-2.56711 \mathrm{E}+02$ | $-2.44827 \mathrm{E}+01$ | $-7.56241 \mathrm{E}+02$ | $-2.95362 \mathrm{E}-012$ |
| 12 | $-9.12949 \mathrm{E}-03$ | $-1.71063 \mathrm{E}+02$ | $-1.66154 \mathrm{E}+01$ | $-5.04159 \mathrm{E}+02$ | $-1.49617 \mathrm{E}-011$ |

Table 3.3: Relative errors for a planar configuration of satellites and the determinant det of matrix used for calculating space-time position.

## Speed of calculations

The simulation was tested on a PC with an Intel Core2 Quad CPU Processor Q6600 at 2.4 GHz and 4GB RAM. The OS was Linux x86_64 with kernel 2.6.28-18-generic, with the Intel Fortran Compiler 10.1. The maximum number of time-steps is $n_{\max }=434$, equivalent to more than one orbit of the satellites. The calculation for steps 1 to 3 described above repeated 434 times (i.e. for 434 time-steps) takes 67.6 seconds to execute ( 0.1558 seconds/time-
step). The time of calculation for steps 2 and 3 (repeated 434 times) is 26.6 seconds ( 0.0682 seconds/time-step). These two steps are actually the ones that the users will perform to determine their positions.

The simulation was also tested on a laptop with an Intel Core2 Duo CPU Processor P8600 at 2.4 GHz and 4GB RAM. The OS is the same as before, but with the Intel Fortran Compiler 11.1. In this case, some processor-specific compiler optimizations are enabled. The maximum number of time-steps is the same as before: $n_{\max }=434$. The time of calculation for steps 1-3 described above repeated 434 times (i.e. for 434 time-steps) is 61.9 seconds ( 0.1426 seconds/time-step). The time of calculation for steps 2 and 3 (repeated 434 times) is 26.5 seconds ( 0.0588 seconds/time-step).

In both cases, a 25 - 32-digit accuracy is achieved when determining the space-time position of the observer.

## NON-GRAVITATIONAL PERTURBATIONS

The important non-gravitational perturbations are those governed by stochastic noise, i.e. by phenomena that cannot be predicted in advance. Clock noise, solar radiation pressure, solar wind pressure and collisions with interplanetary dust particles are sources of such noise. The nature of these forces and also the nature of the uncertainty in knowing their magnitude and direction is quite different.

Clock errors affect the position determination by giving false information on the time of flight from the satellite to the observer. Assuming that a typical clock error can be described as flicker noise with a standard deviation error of $1 \mathrm{~ns} /$ day, we can assign a displacement error of $d_{\mathrm{t}}=0.3 \mathrm{~m}$ per day to stochastic clock perturbations.

Solar radiation pressure produces a force

$$
\begin{equation*}
\vec{F}_{r p}=(1+(\underline{\underline{\eta}}-1) a) \cdot \frac{L_{\odot}}{4 \pi r^{3} c} \vec{r} \mathcal{A} \tag{4.1}
\end{equation*}
$$

with $\vec{r}$ the radius vector from the Sun to the satellite, $\mathcal{A}$ the cross section of the satellite from the impinging direction, $a$ the effective albedo of the satellite (generally depending on the angle of incidence of solar light on the satellite) and $\underline{\underline{\eta}}$ a tensor attached to the satellite, measuring the effective momentum of reflected light. If $a, \underline{\underline{\eta}}$ and $\mathcal{A}$ were constants, the effect could be calculated very precisely, since the solar luminosity is a well known constant. In this case the main effect of solar radiation pressure is a precession of orbital angular momentum with angular velocity

$$
\begin{equation*}
\vec{\Omega}_{r p}=\frac{1}{m l} \vec{l} \times \vec{F}_{r p} \frac{\Omega_{\mathrm{year}}}{\omega_{s}^{2} r_{s}} \tag{4.2}
\end{equation*}
$$

where $\Omega_{\text {year }}$ is the angular velocity of the solar radiation force with respect to the orbital angular velocity of a satellite, i.e. $\Omega_{\text {year }} \simeq \frac{2 \pi}{1 \text { year }}, \omega_{s}=\frac{2 \pi}{P_{s}}\left(P_{s} \sim \frac{1}{2}\right.$ day $)$ is the orbital angular velocity of the satellite and $r_{s}$ is the distance of the satellite from the center of the Earth. This effect, schematically shown in Fig. 4.1, makes the orbital angular momentum oscillate with the orbital period of the Earth around the Sun. The displacement of the satellite's


Figure 4.1: As the Earth (blue sphere) moves about the Sun (yellow sphere), solar radiation pressure makes satellite's orbital angular momentum precess so that the tip of the orbital angular momentum follows the red ellipse; the gray circle showing where the tip of angular momentum would be with no precession.
position due to solar radiation pressure as a function of time can be written as

$$
\begin{equation*}
d_{r p}=\frac{F_{r p}}{4 \pi^{2} m} P_{s}^{2} \sin \left(\Omega_{\mathrm{year}} t\right) \cos \left(\omega_{s} t+\delta\right) \tag{4.3}
\end{equation*}
$$

The amplitude of this term can be considered as a measure of solar radiation strength. If we assume the satellite to have a mass of 600 kg , the cross sectional area of $0.5 \mathrm{~m}^{2}$ and take $\mathcal{F}_{\odot}=1300 \mathrm{~W} / \mathrm{m}^{2}$ for the solar constant, we obtain $d_{r p} \sim 0.17 \mathrm{~m}$. The realistic effect of solar radiation pressure is somewhat more complicated, since $a, \underline{\underline{\eta}}$ and $\mathcal{A}$ depend on the orientation of the satellite with respect to the direction to the Sun. It is probably safe to assume that this dependence can be known to better than $1 \%$ or maybe even $0.1 \%$ in which case the unpredictable stochastic part of the effect of solar radiation pressure would be always below milimeters.

The perturbing force due to solar wind pressure is similar to the radiation pressure force, since it originates from the same direction, only the pressure term $\frac{L_{\odot}}{4 \pi r^{3} c} \vec{r}$ must be replaced by $\frac{\dot{M}}{4 \pi r^{2}} \vec{v}$, where $\dot{M}$ and $\vec{v}$ are solar wind mass loss rate and velocity. Their typical values found in literature are: $\dot{M} \sim 4 \times 10^{19} \mathrm{~g} /$ year and $v \sim 500 \mathrm{~km} / \mathrm{sec}$. Using these numbers we deduce that solar wind pressure is about 200 times weaker than solar radiation pressure. Because of this weakness it may not be important at this point to be able to determine the uncertainty in the solar wind albedo and the equivalent of $\eta$. It is certainly not as important as finding the truly stochastic part of solar wind, which is the force of coronal mass ejections hitting the satellite. Coronal mass ejections are violent outbursts from the solar surface, occurring every few days and carrying away masses $10^{14}$ to $10^{16} \mathrm{~g}$ at speeds up to $5000 \mathrm{~km} / \mathrm{sec}$ (Landi et al. 2010; Zhang et al. 2010). We calculate the momentum received by the satellite ( $\delta p$ ) from impinging material by assuming that the coronal mass flow is spread evenly in a spherical
shell moving away from the Sun*. From this we obtain the velocity change of the satellite $(\delta v=\delta p / m)$, so that the displacement after one day follows: $d_{c m e}=\frac{\delta p}{m} 1$ day. Using the largest numbers quoted, we obtain an estimate $d_{\text {cme }} \sim 0.0003 \mathrm{~m}$ for the purely stochastic part of solar wind displacement a day after the coronal mass ejection.

A relatively small non-gravitational perturbation, but completely stochastic in nature is produced by collisions of satellites with interplanetary dust. Our estimates of these perturbations are based on data from the article (Nesvorny et al. 2006), which studies dust bands Karin and Veritas (Nesvorný et al. 2006). The authors claim that these dust bands contribute from 30 to $50 \%$ of all interplanetary dust in the near Solar system. They contribute 15000 to 20000 tons per year in dust accretion rate to the whole Earth. We assume that in the vicinity of the Earth the dust accretion is aproximately isotropic and that it is proportional to the accreting area $\left(4 \pi R^{2}=4 \mathcal{A}=2 \mathrm{~m}^{2}\right)$. Taking 36000 tons per year as the total mass accretion rate on the Earth, we thus estimate the dust accretion rate per satellite to $1.4 \times 10^{-4} \mathrm{~g} /$ year. In this way we come to the following main conclusions: solar system dust presents a very mild drag resistance to satellite motion around the Earth. The typical orbital decay time scale is of the order of $\sim 3 \times 10^{9}$ years. The stochastic component is contributed mostly by collisions with $100-200 \mu \mathrm{~m}$ dust particles which move with respect to the Earth (and with respect to satellites) with the average velocity $\sim 17 \mathrm{~km} / \mathrm{sec}$ and occur with a probability of less than one per year. The velocity change due to such a collision is $\delta v_{\text {dust }} \sim 5-20 \times 10^{-8} \mathrm{~m} / \mathrm{sec}$. The displacement of a satellite due to such a collision observed after one day would amount to $d_{\text {dust }} \simeq 5-20 \mathrm{~mm}$. Note that such collisions could be detected and their impact measured if satellites would monitor their mutual positions, since the probability for more than one collision to occur during the same day is so small.

It is difficult to compare the strength of these so different non-gravitational perturbations without more extensive data on their character, changing strength and without carefully considering statistical properties of these noise sources. Such a study would go beyond the scope of this report and may not even be very useful before it could be verified experimentally. To make a very rough comparison of importance of each of the listed perturbations, we assign to each perturbation a "force" and a displacement/day. As the worst case we simply use the full perturbing force to calculate the displacement of a satellite after one day. For example we use $F_{r p}=\frac{L_{\odot}}{4 \pi r^{2} c} \mathcal{A}$ for the force of solar radiation pressure and assign it the displacement as $d=\frac{1}{2} \frac{F_{r p}}{m} t^{2}(t=1$ day $)$. As explained above, this is a gross overestimate, at least in the case of solar radiation pressure or solar wind pressure. We attempt to refine such noise upper limit by estimating only the stochastic part of 1-day position error, i.e. the error remaining after the known part of the perturbing force has been taken into account with

[^4]best available data. Our estimates are given in Table 4.1. The reader should be aware of

Table 4.1: Nongravitational noise sources

| perturbation | magnitude | displacement/day | stochastic component |
| :---: | :---: | :---: | :---: |
| clock noise | $1 \mathrm{~ns} /$ day | - | 0.3 m |
| radiation pressure | $4.3 \times 10^{-6} \mathrm{Nm}^{-2}$ | 13 m | $<0.001 \mathrm{~m} ?$ |
| solar wind pressure | $2.2 \times 10^{-9} \mathrm{Nm}^{-2}$ | 0.007 m | $\sim 0.0003 \mathrm{~m} ?$ |
| dust collisions | $\sim 100 \mu \mathrm{~m}$ part | - | $\sim 0.00005 \mathrm{~m}$ |

the large uncertainty involved in these estimates.
Dust collision differ from all other perturbations considered by their very low probability rate. Their per day displacement in Table 4.1 is thus just a very long term average. A more detailed table of collision probabilities and displacements, in this case per year, is listed in Table 4.2.

Table 4.2: Dust particle collision probabilities

| size $[\mu \mathrm{m}]$ | collisions per year | displacement/year $[\mathrm{m}]$ |
| :---: | :---: | :---: |
| $67-114$ | 0.67 | $0.8-4$ |
| $114-144$ | 0.35 | $4-8$ |
| $144-166$ | 0.17 | $8-12$ |
| $166-182$ | 0.08 | $12-16$ |
| $182-196$ | 0.05 | $16-20$ |
| $>196$ | 0.17 | $>20$ |

Even if the above estimates are very rough, it is clear that timing noise is by far the most important noise source that must be controlled on a longer time scale by other means as for example telemetry from the ground. It is worth pointing out, however, that the nongravitational perturbations, apart from the slow drag mentioned above, do not systematically change orbital momentum of satellites (after all the precessions due to other gravitational perturbations have been properly taken into account). This means that the area swept by the radius vector from the center of the Earth to each satellite can be a clock with very good long term phase stability limited in principle only by the uncertainty in swept area. It would be possible to use the timing capability of the constellation of satellites, if each satellite was capable of detecting the timing signal of all other visible satellites and use it, just like any other observer, to determine its own space-time position.

## Conclusion

In this study we explore the possibility of using the global navigation system to map the space-time around the Earth, as well as the possibility that such an approach could improve the accuracy and longer range autonomy of the global positioning system. The study focuses on the use of general relativistic null-coordinates as natural observer independent coordinates which are automatically realized by radio wave communications from satellites to GNSS users and by radio communications between satellites themselves.

The use of null-coordinates is realized in the local Schwarzschild coordinate system with three spatial orthonormal directions $\underline{e}_{i}$ and one time-oriented vector $\underline{e}_{0}$, with coordinates $X, Y, Z, t$. Before taking into account gravitational perturbations due to the Moon, Sun, planets, obliquity and rotation of the Earth ..., this local coordinate system can also be considered as a global inertial system, i.e. oriented in fixed directions with respect to distant stars. Since the main effect of gravitational perturbations is to make the local Schwarzschild frame precess about different axes determined by orbits of perturbing bodies and these precessions are very slow, the local Schwarzschild coordinate frame can to a high degree of accuracy be considered as inertial (Precessions are also accompanied by much faster nutations, however with much smaller amplitudes, amounting to only a few meters). Due to this fact it is possible to decouple the problem of motion in the local Schwarzschild frame from the problem of connecting the local Schwarzschild frame to the global inertial frame. This second problem is well understood in the framework of classical non-relativistic gravitational perturbation theory, but is not the subject of this study. Here we concentrated on finding algorithms to describe the dynamics ob bodies in the local Schwarzschild frame in full general relativity, in defining null-coordinates in space-time that are tied to the constellation of satellites and in reading these coordinates in order to determine the Schwarzschild coordinates of an event in space-time. Thus, we have tackled the tasks defined at the negotiation meeting and obtained the following results:

- Analytic solutions for light-like (open) and time-like (closed) geodesics were derived and cast into a form suitable for calculations in the limit of weak gravitational fields. The transition of these expressions to the newtonian limit has also been checked and verified numerically. It has been confirmed that in the case of Galileo satellites relative
differences between classical and relativistic solutions amount to relative differences of the order of $10^{-9}$ per orbit, which is consistent with post-newtonian predictions.
- We defined two algorithms:
a) an algorithm that calculates null-coordinates $\left(\tau_{1}, \tau_{2}, \tau_{3}, \tau_{4}\right)$ corresponding to local Schwarzschild coordinates $\left(t_{o}, x_{o}, y_{o}, z_{o}\right)$ of an observer (or a receiving satellite),
b) and the "reverse" algorithm that calculates space-time coordinates $\left(t_{o}, x_{o}, y_{o}, z_{o}\right)$ of an observer (or a receiving satellite) from its null-coordinates $\left(\tau_{1}, \tau_{2}, \tau_{3}, \tau_{4}\right)$.
- For proof of concept the two algorithms have been first written in Mathematica. These codes have a compact logical structure that makes them relatively simple to read, but are too slow in execution and are thus not suitable for extended cross tests.
- The algorithm (a) has also been written in Fortran and $\mathrm{C}++$. The two codes have been tested against each other. The algorithm (b) has been written in Fortran too and checked against the first algorithm for consistency. All codes have been optimized for weak gravitational fields. The codes have been tested for speed on a PC with a quad-core CPU, and a laptop with a dual-core CPU. It takes $\sim 0.06-0.7$ seconds to determine all four coordinates at 25 - 32-digit precision. This makes our codes feasible and could be used in modern positioning devices.
- Effects of non-gravitational perturbations have been studied. We focused mainly on the stochastic part of these perturbations. Clock noise has been identified as the most important contributor affecting the accuracy of position determination. By magnitude the second perturbing force is the solar radiation pressure. We note, however, that this force is very stable and could, at least in principle, be modeled to very high precision. Therefore, its stochastic effect is not expected to contribute significantly more to position noise than the remaining two truly stochastic perturbations: solar wind pressure and collisions with interplanetary dust. We also note that stochastic forces have no preferred direction, therefore they are not expected to change average values of orbital angular momenta of satellites. This means that areal velocities of satellites are good constants of motion perturbed only by deterministic gravitational perturbations. Therefore we suggest that the area covered by the radius vector from the center of the Earth to satellites be used as a stable measure of proper time of this satellite.

We believe to have shown that the use of fully relativistic code in GNSS systems offers an interesting alternative to using post-Newtonian approximations. The code written and tested executes a position finding algorithm in about 60 ms on a normal laptop computer
if the initial position is completely unknown and a few times faster if the algorithm starts from a tighter dead reckoning position. This proves that the more consistent relativistic formalism presents no technical obstacle for use in modern global positioning devices. The advantages of using a fully relativistic code over the use of classical post-newtonian codes go well beyond aesthetics. Since stochastic perturbations on the constellation of satellites are small, the constellation can be given greater autonomy in defining its own local Schwarzschild frame in which the predictability of dynamics can be taken advantage of. In particular the long term clock drifts can be corrected by using orbital area as a uniformly increasing proper time coordinate. In this system the angular velocities of slow precessions between the local Schwarzschild frame and the global inertial frame give precise data on the local space-time metric.

Finally, we would like to argue that the concept of autonomy of the constellation of satellites, that emerged from this study, deserves further attention. It offers increased accuracy and long term stability to the global positioning system, as well as promises us a tool for a systematic study of the metric of the space-time around the Earth, which can also give us a deeper insight into what is going on inside the Earth. In order to realize such an autonomous constellation, it must be self-consistent. By this we mean that if satellites determine their orbits by using positioning data from other satellites in the constellation, they should obtain the same constants of motion that were used in the definition of the global positioning system. Such self-consistency at the maximum level of precision is not obvious, but can be reached if each satellite is also a receiver of timing signals of all other satellites. We believe we can design a mathematical procedure to drive any initial GNSS solution toward the best self consistent solution.

## Equations in SchwarzSchild SPACE-TIME

Starting with the results of Chandrasekhar (1992) and Rauch and Blandford (1994), who expressed the solutions to geodesic equations in terms of elliptic integrals and by inverting their expressions into Jacobi elliptic functions (Čadež et al. 1998), simpler solutions of geodesic equations are obtained (Gomboc 2001). Such solutions, which no longer contain the branch ambiguity, are presented here.

The solutions exist only as long as $P(u) \geq 0$, where $P(u)=A^{2}-u^{2}(1-u)+B(1-u)$ is the polynomial in (2.4), and depend on the three roots of $P(u)$. The discriminant $D$, which is defined as:

$$
\begin{align*}
\alpha & =1-9 B-\frac{27}{2} A^{2}  \tag{A.1}\\
\beta & =-1-3 B  \tag{A.2}\\
D & =\alpha^{2}+\beta^{3}, \tag{A.3}
\end{align*}
$$

determines the nature of the roots of $P(u)$. Since the orbits extend at most from $u=0$ to $u=1$, only the roots on this interval are of interest. The roots and four possible orbit types are shown in Fig. A.1.

A similar overview of orbits can be obtained by comparing the energy $E$ against the effective potential $V$ (Misner et al. 1973), since the orbits can exist only if $E \geq V$. For time-like geodesics, the effective potential $V$ is defined as:

$$
\begin{equation*}
V=\sqrt{(1-u)\left(1+\tilde{l}^{2} u^{2}\right)} \tag{A.4}
\end{equation*}
$$

and depends on the reduced angular momentum $\tilde{l}=l / 2 M$. The orbit types for $\tilde{l}=2.2$ and different values of $E$ are shown in Fig. A. 2 (left). Since light-like geodesics depend only on


Figure A.1: The polynomial $P(u)$ and the distribution of its roots in the interval $u \in[0,1]$. The orbits exist only where $P(u) \geq 0$, which is shown in green. The corresponding orbit types are marked with letters $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ or $\boldsymbol{D}$ and the roots are marked with $u_{1}, u_{2}$ and $u_{3}$. The sign of the discriminant $D$ is also noted.


Figure A.2: Left: The effective potential $V$ for time-like geodesics (Eq. A.4) with the value of reduced angular momentum of $\tilde{l}=2.2$. By choosing appropriate value of $E$, orbits of any type can be constructed. Right: The effective potential $V / \tilde{l}$ for light-like geodesics (Eq. A.5). Since the potential has no minimum, only orbits of type $\boldsymbol{A}, \boldsymbol{B}$ or $\boldsymbol{C}$ can be constructed by choosing an appropriate value of $A$. In case of light-like geodesics the critical orbits are also shown. Such orbits are orbits with $E=V_{\max }$, where $V_{\max }$ is the maximum of the effective potential. Note, that for $V_{\min } \leq E \leq V_{\max }$ the discriminant is $D \leq 0$, and $D>0$ otherwise.
parameter $A=E / \tilde{l}(b=0)$, the effective potential $V$ does not depend on any parameters:

$$
\begin{equation*}
\frac{V}{\tilde{l}}=\sqrt{u^{2}(1-u)} . \tag{A.5}
\end{equation*}
$$

The orbit types for photons and different values of $A$ are shown in Fig. A. 2 (right). Here it can be seen again that photons cannot be on bound orbits because the effective potential only has a maximum and no minimum. Note, that a similar situation can also occur for massive particles if $\tilde{l}<\sqrt{3}$. However, since both the minimum and the maximum of $V$ for time-like orbits disappear, only the orbits of type $\boldsymbol{B}$ and $\boldsymbol{C}$ exist in this case.

Exact solutions to Eqs. (2.4) - (2.8b) for orbit types that are applicable to the Galileo GNSS are given in the following sections.

## A. 1 Time-like geodesics

The four types of orbits for massive particles have the following properties:

- type $\boldsymbol{A}$ : scattering orbits with both endpoints at infinity. Scattering orbits can never extend below $r=3 M$.
- type $B$ : plunging orbits with one end at infinity and the other behind the horizon,
- type $C$ : near orbits with both ends behind the horizon of the black hole. Near orbits can never reach above $r=6 M$.
- type $\boldsymbol{D}$ : closed orbits. Highly eccentric orbits can never reach below $r=4 M$ while circular orbits can never reach below $r=6 M$.

Since GNSS satellites are on type $\boldsymbol{D}$ orbits, only the solutions for this type are presented here.

For easier comparison with classical orbits, the orbital parameters $\eta=-1+E / m_{\oplus} c^{2}$ (orbital energy) and $\tilde{l}=l /\left(2 m_{\oplus} r_{\mathrm{g}} c\right)$ (areal velocity) are expressed with the Keplerian parameters $a$ (major semi-axis) and $\varepsilon$ (eccentricity) of the orbit* such that

$$
\begin{align*}
\tilde{l} & =\frac{1}{2} \sqrt{a\left(1-\varepsilon^{2}\right)}  \tag{A.6}\\
\eta & =-\frac{1}{2 a}, \tag{A.7}
\end{align*}
$$

[^5]However, note that $a$ and $\varepsilon$ defined in this way do not correspond precisely to the Schwarzschild $a_{S}$ and $\varepsilon_{S}$ defined as:

$$
\begin{align*}
a_{S} & =\frac{1}{u_{2}}+\frac{1}{u_{3}}  \tag{A.8}\\
\varepsilon_{S} & =\frac{u_{2}-u_{3}}{u_{2}+u_{3}} \tag{A.9}
\end{align*}
$$

The final choice of most appropriate orbital parameters can be left open until technical implementation of these solutions.

Introduce additional parameters:

$$
\begin{align*}
q & =\frac{3 \sqrt{12 \eta^{2}+\left(-81 \eta^{4}-324 \eta^{3}-378 \eta^{2}-108 \eta+3\right) \tilde{l}^{-2}+24 \eta-12 \tilde{l}^{-4}}}{2 \tilde{l}\left(1-3 \tilde{l}^{-2}\right)^{3 / 2}}  \tag{A.10a}\\
\psi_{a} & =-\arcsin (q)  \tag{A.10b}\\
p & =\sqrt{1-3 / \tilde{l}^{2}}  \tag{A.10c}\\
U_{1} & =p+\frac{1 / \tilde{l}^{2}}{1+p}-\frac{4 p}{3} \sin ^{2} \frac{\psi_{a}}{6}  \tag{A.10d}\\
U_{2} & =\frac{1}{3}\left(\frac{3 / \tilde{l}^{2}}{1+p}-p \sqrt{3} \sin \frac{\psi_{a}}{3}+2 p \sin ^{2} \frac{\psi_{a}}{6}\right)  \tag{A.10e}\\
U_{3} & =\frac{1}{3}\left(\frac{3 / \tilde{l}^{2}}{1+p}+p \sqrt{3} \sin \frac{\psi_{a}}{3}+2 p \sin ^{2} \frac{\psi_{a}}{6}\right)  \tag{A.10f}\\
n_{a} & =\frac{1}{\sqrt{U_{1}-U_{3}}}  \tag{A.10g}\\
m_{a} & =\frac{U_{2}-U_{3}}{U_{1}-U_{3}}  \tag{A.10h}\\
n_{a 1} & =1-\frac{U_{2}}{U_{3}}  \tag{A.10i}\\
n_{a 2} & =\frac{U_{2}-U_{3}}{1-U_{3}} . \tag{A.10j}
\end{align*}
$$

Here $U_{1}, U_{2}$, and $U_{3}$ correspond to the roots of the polynomial $P(u)$ shown in Fig. A. 2 (top). All analytical expressions have been optimized for a weak gravitational field ( $\psi_{a} \ll 1$ ). We advise to manually derive the expression $\left(U_{2}-U_{3}\right)$ from $U_{2}$ and $U_{3}$, and use it in $m_{a}, n_{a 1}$, and $n_{a 2}$ to avoid precision loss. With these parameters, the equations (2.4) - (2.8b) are solved to


Figure A.3: Time-like geodesics of type $\boldsymbol{D}$. Left: An orbit with $\tilde{l}=2.32379$ and $E=0.986$. Middle: Innermost stable circular orbit (ISCO) with $\tilde{l}=\sqrt{3}$ and $E=2 \sqrt{3} / 3$. Right: Orbit of the SO-16 star near Sgr A* with $\tilde{l}=25.196211749272674$ and $E=0.999989536192044$. The values of $\tilde{l}$ and $E$ are calculated from Ghez et al. (2005). In the first two examples, the radius of the black circle is the Schwarzschild radius. In the last example only the position of the black hole is marked by ' $x$ '.
get the type $\boldsymbol{D}$ orbit, time and proper time:
$u(\chi)=U_{2}-\left(U_{2}-U_{3}\right) \cos ^{2} \chi$
$t(\chi)=\frac{1+\eta}{\tilde{l}} \frac{4 n_{a}}{U_{3}^{2}}\left[\left(1+U_{3}+\frac{n_{a 1}^{2}-m_{a}}{2\left(m_{a}-n_{a 1}\right)\left(n_{a 1}-1\right)}\right) \Pi\left(n_{a 1} ; \chi \mid m_{a}\right)+\frac{U_{3}^{2}}{1-U_{3}} \Pi\left(n_{a 2} ; \chi \mid m_{a}\right)\right.$
$\left.+\frac{n_{a 1} / 2}{\left(m_{a}-n_{a 1}\right)\left(n_{a 1}-1\right)}\left(\mathrm{E}\left(\chi \mid m_{a}\right)-\left(1-\frac{m_{a}}{n_{a 1}}\right) \mathrm{F}\left(\chi \mid m_{a}\right)-\frac{n_{a 1} \sin 2 \chi \sqrt{1-m_{a} \sin ^{2} \chi}}{2\left(1-n_{a 1} \sin ^{2} \chi\right)}\right)\right]$
$\tau(\chi)=\frac{t(\chi)}{1+\eta}-\frac{4 n_{a}}{U_{3} \tilde{l}}\left(\Pi\left(n_{a 1} ; \chi \mid m_{a}\right)+\frac{U_{3}}{1-U_{3}} \Pi\left(n_{a 2} ; \chi \mid m_{a}\right)\right)$,
where the parameter $\chi$ and true anomaly $\lambda$ are related in the following way:

$$
\begin{align*}
& \chi(\lambda)=\operatorname{am}\left(\left.\mathrm{K}\left(m_{a}\right)+\frac{\lambda}{2 n_{a}} \right\rvert\, m_{a}\right)  \tag{A.14}\\
& \lambda(\chi)=2 n_{a}\left(\mathrm{~F}\left(\chi \mid m_{a}\right)-\mathrm{K}\left(m_{a}\right)\right) \tag{A.15}
\end{align*}
$$

Both parameters can have values in the interval $\lambda \in(-\infty, \infty)$ and $\chi \in(-\infty, \infty)$. The values of $\chi$ at periapsis and apoapsis are $\pi / 2$ and 0 respectively. The definitions of the elliptic integrals and functions can be found in App. C.

From equations (A.11) and (A.14) it is straightforward to obtain the following form of the orbit equation (2.9)

$$
\begin{equation*}
u(\lambda)=U_{2}-\left(U_{2}-U_{3}\right) \mathrm{cn}^{2}\left(\left.\mathrm{~K}\left(m_{a}\right)+\frac{\lambda}{2 n_{a}} \right\rvert\, m_{a}\right) \tag{A.16}
\end{equation*}
$$

Few examples of type $\boldsymbol{D}$ orbits are shown in Fig. A.3.

## A. 2 Light-like geodesics

The orbit equations for photons depend only on the parameter $A$ and are of three types:

- type $\boldsymbol{A}$ : scattering orbits with both endpoints at infinity; their angular momentum parameter is on the interval $0<A<\frac{2}{3 \sqrt{3}}$. Scattering orbits can never extend below $r=3 M$.
- type $B$ : plunging orbits with one end at infinity and the other behind the horizon; $A>\frac{2}{3 \sqrt{3}}$,
- type $C$ : near orbits with both ends behind the horizon of the black hole; their angular momentum parameter is on the interval $0<A<\frac{2}{3 \sqrt{3}}$. Near orbits can never reach beyond $r=3 M$.

Since we are interested in the interlinks between GNSS satellites, only the solutions for type $\boldsymbol{A}$ orbits are presented here.*

Introduce the following auxiliary parameters:

$$
\begin{align*}
\psi & =2 \arcsin \left(\frac{3 \sqrt{3}}{2} A\right)  \tag{A.17a}\\
u_{1} & =1-\frac{4}{3} \sin ^{2} \frac{\psi}{6}  \tag{A.17b}\\
u_{2} & =\frac{2}{3} \sin ^{2} \frac{\psi}{6}+\frac{1}{\sqrt{3}} \sin \frac{\psi}{3}  \tag{A.17c}\\
u_{3} & =\frac{2}{3} \sin ^{2} \frac{\psi}{6}-\frac{1}{\sqrt{3}} \sin \frac{\psi}{3}  \tag{A.17d}\\
m & =\frac{2 \tan \frac{\psi}{3}}{\tan \frac{\psi}{3}+\sqrt{3}}  \tag{A.17e}\\
n & =\frac{\sqrt{1-2 \sin ^{2} \frac{\psi}{6}+\frac{1}{\sqrt{3}} \sin \frac{\psi}{3}}}{\sqrt{1-\frac{u_{2}}{u_{3}}}}  \tag{A.17f}\\
n_{1} & =1  \tag{A.17g}\\
n_{2} & =\frac{u_{2}-u_{3}}{1-u_{3}} \tag{A.17h}
\end{align*}
$$

[^6]

Figure A.4: Some orbits of type $A$ with initial point at $r_{i}=20 M$ and parameters $\frac{2}{3 \sqrt{3}}-A \in$ $\{0.2,0.15,0.1,0.05,0.005\}$. The radius of the black circle is the Schwarzschild radius.

Here $u_{1}, u_{2}$, and $u_{3}$ correspond to the roots of the polynomial $P(u)$ shown in Fig. A. 2 (top). All analytical expressions have been optimized for weak gravitational fields $(\psi \ll 1)$. We advise to manually derive the expression $\left(u_{2}-u_{3}\right)$ from $u_{2}$ and $u_{3}$, and use it in $n_{1}$ and $n_{2}$ to avoid precision loss. With these parameters, the equations (2.4) - (2.8a) are solved to get the type $\boldsymbol{A}$ orbit and time:

$$
\begin{align*}
u(\chi)= & u_{2}-\left(u_{2}-u_{3}\right) \cos ^{2} \chi  \tag{A.18}\\
t(\chi)= & \frac{2 n}{u_{3}^{2}} \frac{2}{3 \sqrt{3}} \sin \frac{\psi}{2}\left[\left(1+u_{3}+\frac{n_{1}^{2}-m}{2\left(m-n_{1}\right)\left(n_{1}-1\right)}\right) \Pi\left(n_{1} ; \chi \mid m\right)+\frac{u_{3}^{2}}{1-u_{3}} \Pi\left(n_{2} ; \chi \mid m\right)\right. \\
& \left.+\frac{n_{1} / 2}{\left(m-n_{1}\right)\left(n_{1}-1\right)}\left(\mathrm{E}(\chi \mid m)-\left(1-\frac{m}{n_{1}}\right) \mathrm{F}(\chi \mid m)-\frac{n_{1} \sin 2 \chi \sqrt{1-m \sin ^{2} \chi}}{2\left(1-n_{1} \sin ^{2} \chi\right)}\right)\right] \tag{A.19}
\end{align*}
$$

where the parameter $\chi$ and true anomaly $\lambda$ are related in the following way:

$$
\begin{align*}
& \chi(\lambda)=\operatorname{am}\left(\left.\mathrm{K}(m)+\frac{\lambda}{n} \right\rvert\, m\right)  \tag{A.20}\\
& \lambda(\chi)=n(\mathrm{~F}(\chi \mid m)-\mathrm{K}(m)) . \tag{A.21}
\end{align*}
$$

The parameter $\chi$ can have values in the interval $\chi \in\left(\chi_{\min }, \chi_{\max }\right)$, where $\chi_{\min }=$ $\arccos \left(\sqrt{u_{2} /\left(u_{2}-u_{3}\right)}\right)$ and $\chi_{\text {max }}=\arccos \left(-\sqrt{u_{2} /\left(u_{2}-u_{3}\right)}\right)$. The true anomaly $\lambda$ can have values in the interval $\lambda \in\left(\mathrm{F}\left(\chi_{\max } \mid m\right)-\mathrm{K}(m), \mathrm{F}\left(\chi_{\min } \mid m\right)-\mathrm{K}(m)\right)$. The value of $\chi$ at periapsis is $\pi / 2$. The definitions of the elliptic integrals and functions can be found in App. C.

From equations (A.18) and (A.20) it is straightforward to obtain the following form of the orbit equation (2.10)

$$
\begin{equation*}
u(\lambda)=u_{2}-\left(u_{2}-u_{3}\right) \mathrm{cn}^{2}\left(\left.\mathrm{~K}(m)+\frac{\lambda}{n} \right\rvert\, m\right) \tag{A.22}
\end{equation*}
$$

Few examples of type $\boldsymbol{A}$ orbits are shown in Fig. A.4.

## A. 3 Ray-tracing

The method for determining the orbital parameter $A$ has been described in ČK05, but to apply it in the weak gravitational field limit, some care must be taken to avoid numerical cancellation errors. For this purpose, the RHS of equation (19) from ČK05, i.e. equations (20) and (21) are rewritten in the following way:

$$
\begin{align*}
\text { right }_{1} & =-2 \sqrt{3}\left(\left(3 \cos \frac{\psi}{3}+\sqrt{3} \psi_{3}\right) \sqrt{\left(3 u_{f}-2 \psi_{6}-\sqrt{3} \psi_{3}\right)\left(3 u_{i}-2 \psi_{6}-\sqrt{3} \psi_{3}\right)}\right. \\
& \left.+\sqrt{\left(3-3 u_{f}-4 \psi_{6}\right)\left(3-3 u_{i}-4 \psi_{6}\right)\left(3 u_{f}-2 \psi_{6}+\sqrt{3} \psi_{3}\right)\left(3 u_{i}-2 \psi_{6}+\sqrt{3} \psi_{3}\right)}\right) / \\
& \left(18 \sqrt{3} u_{i} u_{f}-4 \sqrt{3} \psi_{6}\left(2+3\left(u_{f}+u_{i}\right)+4 \cos \frac{\psi}{3}\right)-12 \psi_{3}+18 \psi_{3}\left(u_{i}+u_{f}\right)-12 \sin \frac{2 \psi}{3}\right) \tag{A.23}
\end{align*}
$$

$$
\begin{align*}
\text { right }_{2} & =-2 \sqrt{3}\left(-\left(3 \cos \frac{\psi}{3}+\sqrt{3} \psi_{3}\right) \sqrt{\left(3 u_{f}-2 \psi_{6}-\sqrt{3} \psi_{3}\right)\left(3 u_{i}-2 \psi_{6}-\sqrt{3} \psi_{3}\right)}\right. \\
& \left.+\sqrt{\left(3-3 u_{f}-4 \psi_{6}\right)\left(3-3 u_{i}-4 \psi_{6}\right)\left(3 u_{f}-2 \psi_{6}+\sqrt{3} \psi_{3}\right)\left(3 u_{i}-2 \psi_{6}+\sqrt{3} \psi_{3}\right)}\right) / \\
& \left(18 \sqrt{3} u_{i} u_{f}-4 \sqrt{3} \psi_{6}\left(2+3\left(u_{f}+u_{i}\right)+4 \cos \frac{\psi}{3}\right)-12 \psi_{3}+18 \psi_{3}\left(u_{i}+u_{f}\right)-12 \sin \frac{2 \psi}{3}\right), \tag{A.24}
\end{align*}
$$

where $\psi_{3}$ and $\psi_{6}$ are defined as

$$
\begin{align*}
& \psi_{3}=\sin \frac{\psi}{3}  \tag{A.25a}\\
& \psi_{6}=\sin ^{2} \frac{\psi}{6} \tag{A.25b}
\end{align*}
$$

and $u_{i}=2 M / r_{i}$ and $u_{f}=2 M / r_{f}$ are the radial coordinates of the initial and the final point respectively. The LHS of equation (19) from ČK05 remains the same:

$$
\begin{equation*}
\text { left }=\operatorname{cn}\left(\left.\frac{\Delta \lambda}{n} \right\rvert\, m\right) \tag{A.26}
\end{equation*}
$$

The solution to the equation left $=$ right $_{1}$ or left $=$ right $_{2}$ determines the value of parameter $\psi$, as shown in Fig. A.5. This equation for $\psi$ is solved numerically (using e.g. bisection or Newton method) starting from the zero-order approximation $\psi^{(0)}$ from Kepler:

$$
\begin{equation*}
\psi^{(0)}=3 \arcsin \sqrt{\frac{3}{4}\left(\left(1+\cot \frac{\Delta \lambda}{2}\right)\left(u_{i}-u_{f}\right)^{2}+\left(u_{i}+u_{f}\right)^{2}\left(1+\tan \frac{\Delta \lambda}{2}\right)\right)} . \tag{A.27}
\end{equation*}
$$



Figure A.5: Finding the value of $\psi$ from (A.23) - (A.26) for $\Delta \lambda=168.05^{\circ}, r_{i}=2.67 \times$ $10^{9} M$, and $r_{f}=2 \times 10^{10} M$. The values were chosen to approximate the values for GNSS satellites. The red line marks the numerically found value for $\psi=2.1244 \times 10^{-8}$.

The last step is to determine the time-of-flight between $\mathcal{P}_{i}$ and $\mathcal{P}_{f}$. First, choose as the initial point $u_{i}$ one that is at a greater radius, and the final point $u_{f}$ one that is at a smaller radius:

$$
\begin{align*}
u_{i} & =2 M / \max \left(r_{i}, r_{f}\right)  \tag{A.28}\\
u_{f} & =2 M / \min \left(r_{i}, r_{f}\right) \tag{A.29}
\end{align*}
$$

Next, calculate $\chi_{i}$ at $u_{i}$ and the difference of true anomaly $\Delta \lambda_{p}$ from this point to the periapsis:

$$
\begin{align*}
\chi_{i} & =\arccos \sqrt{\frac{u_{2}-u_{i}}{u_{2}-u_{3}}}  \tag{A.30}\\
\Delta \lambda_{p} & =n\left(\mathrm{~K}(m)-\mathrm{F}\left(\chi_{i} \mid m\right)\right), \tag{A.31}
\end{align*}
$$

where $n$ and $m$ are determined by (A.17f) and (A.17e). The value of $\chi_{f}$ depends on whether the orbit to the final point passes the periapsis or not:

$$
\chi_{f}= \begin{cases}\arccos \left(\sqrt{\frac{u_{2}-u_{f}}{u_{2}-u_{3}}}\right) & \text { if } \Delta \lambda_{p}>\Delta \lambda  \tag{A.32}\\ \arccos \left(-\sqrt{\frac{u_{2}-u_{f}}{u_{2}-u_{3}}}\right) & \text { if } \Delta \lambda_{p}<\Delta \lambda\end{cases}
$$

Finally, use $\chi_{i}$ and $\chi_{f}$ in (A.19) to calculate the time-of-flight:

$$
\begin{equation*}
\Delta t=t\left(\chi_{f}\right)-t\left(\chi_{i}\right) \tag{A.33}
\end{equation*}
$$

# Calculating Schwarzschild COORDINATES FROM NULL-COORDINATES 

In this chapter we give the Mathematica routines used for the calculation of Schwarzschild coordinates of a user, knowing the four times sent by four satellites and their orbital parameters. Each routine could be used separately. These algorithms are "proof-ofconcept", and their execution can be slow. We remind here that they have been written in Fortran for extensive tests. The Fortran codes are publicly available on the website atlas.estec.esa.int/ariadnet.

The input parameters of the Main program are MyTime - dead reckoning time coordinate and vector $\tau$ in the form $\tau=\left\{\left\{N o_{1}, \tau_{1}\right\},\left\{N o_{2}, \tau_{2}\right\},\left\{N o_{3}, \tau_{3}\right\},\left\{N o_{4}, \tau_{4}\right\}\right\}$, where $N o_{1} \ldots N o_{4}$ are the numbers of the four satellites providing their proper time signals $\tau_{1}$ to $\tau_{4}$. The output gives the position of the observer and the calculated error, i.e. the residue of (3.3) in the following form $\left\{\{X, Y, Z, t\},\left\{\delta t_{1}, \delta t_{2}, \delta t_{3}, \delta t_{4}\right\}\right\}$.

The main program needs the following routines:

- Satelit $[k]$ which returns the constants of motion of the $k^{\text {th }}$ satellite in the following order $\left\{\iota, \omega, \Omega, \varepsilon, a, t_{0}\right\}$
- Orbit $[\lambda$, Parameters $]$ returns the position of the satellite with the given orbital parameters (Parameters) as a function of true anomaly $(\lambda)$ in the form: $\{X, Y, Z, t, \tau\}$. Parameters $=\left\{\iota, \omega, \Omega, \varepsilon, a, t_{0}\right\}$.
- Tau[ $\lambda$, Parameters $]$ returns the proper time of the satellite with orbital Parameters if true anomaly is $\lambda$; note: the value of $\lambda$ is limited only by the range of reals defined by the computer.
- Anomaly $[\tau, S a t]$, returns the true anomaly of a satellite with Parameters $=$ Satelit[Sat]
- to $\chi F\left[r_{i}, r_{f}, \theta\right]$ returns the time of flight between two points at $r_{i}$ and $r_{f}$ from the center of the Earth and the view angle of these points from the center of the Earth is $\theta$.
- Iteriraj $\left[t_{1}, t_{2}\right][R a, T o c]$; The input parameter Ra is a vector $\left\{t_{N o_{1}}-t_{m}, t_{N o_{2}}-\right.$ $\left.t_{m}, t_{N o_{3}}-t_{m}, t_{N o_{4}}-t_{m}\right\}$, where $t_{N o_{k}}$ is the global Schwarzschild time at the moment when the $\mathrm{k}^{t h}$ satellite emitted the proper time signal $\tau_{k}$ and $t_{m}$ is the current estimate for the time coordinate of the observer. Toc is the vector of four 3 -vectors $\left\{\left\{X_{N o_{1}}, Y_{N o_{1}}, Z_{N o_{1}}\right\},\left\{X_{N o_{2}}, Y_{N o_{2}}, Z_{N o_{2}}\right\},\left\{X_{N o_{3}}, Y_{N o_{3}}, Z_{N o_{3}}\right\},\left\{X_{N o_{4}}, Y_{N o_{4}}, Z_{N o_{4}}\right\}\right\}$, i.e. positions of satellites when they emitted their timing signals. The input parameters $t_{1}$ and $t_{2}$ specify the interval $\left\{t_{m}+t_{1}, t_{m}+t_{2}\right\}$ on which the solution for the observer time should be searched. The explicit output of Iteriraj is a new narrower interval $\left\{t_{1}, t_{2}\right\}$. Iteriraj also has an implicit output in the vector rez, such that $r e z[[1]]$ and $r e z[[2]]$ are the 3 -coordinates of points on line 1 and line 2 (see text above), where these lines meet at closest distance.
- Matrix[Toc, Pos] calculates the characteristic matrix of (3.6), where Toc are positions of satellites as in Iteriraj and Pos is the best space-time position of the observer before relativistic correction.
- RHS calculates the left hand side of (3.6). Cas $[A]$ takes the 4 -th component of a vector (global Schwarzschild time) and Polozaj[A] makes a position 3 -vector out of a four or 5-vector.


## Main program

```
WhereGalileo =
    Function[{MyTime, \tauin },i1= IntegerPart [\tauin[[1, 1]]];
i2 = IntegerPart [\tauin[[2, 1]]];
i3 = IntegerPart [\tauin[[3, 1]]];
i4 = IntegerPart [\tauin[[4, 1]]];
FourSatellites = {Orbit[Anomaly[\tauin[[1, 2]],Satelit[i1]],Satelit[i1]],
    Orbit[Anomaly[\tauin[[2, 2]], Satelit[i2]], Satelit[i2]],
    Orbit[Anomaly[\tauin[[3, 2]], Satelit[i3]], Satelit[i3]],
    Orbit[Anomaly[\tauin[[4, 2]], Satelit[i4]],Satelit[i4]]};
Toc =Table[Polozaj[FourSatellites[[i]]], {i,4}];
Ra = Table[MyTime - Cas[FourSatellites[[i]]],{i,4}];
TimeInterval = 0.01Sqrt[Toc[[1]].Toc[[1]]];
Kam = Iteriraj[{-TimeInterval,TimeInterval}][Ra,Toc];Sigma = 1;
While[Sigma > 2 < 10-15 Rd1, Kam = Iteriraj[Kam][Ra,Toc]];
SpPos0 = Append[rez[[1]],Avg];
ME = Matrix[Toc, Polozaj[SpPos0]];
Razlika = RHS[Ra,Toc,rez[[1]]] - (Cas[SpPos0]);
X = x, y,z,ttt;
RRes = Solve[ME.X == Razlika, x, y,z,ttt];
\DeltaP=X/.RRes[[1]];
NP=Polozaj[SpPos0] + Table[\DeltaP[[i]],i,3];
RazlikaF = RHS[Ra,Toc,NP]-(Cas[SpPos0]) - \DeltaP[[4]];
RNres = Solve[ME.X == RazlikaF, x,y,z,ttt];
\DeltaPN = X/.RNres[[1]];
NNP=Polozaj[SpPos0] + Table[\DeltaP[[i]] + \DeltaPN[[i]],i,3];
SpaceTimePositionN ={Append[NNP,(Cas[SpPos0])+\DeltaP[[4]]+\DeltaPN[[4]]],
    RHS[Ra,Toc,NNP] - (Cas[SpPos0]) - \DeltaPN[[4]] - \DeltaP[[4]]}];
```


## Routines

Orbit $=$ Function $[\{\phi$, Parameter $s\}$,
$\iota=$ Parameters $[[1]] ; \omega=$ Parameters $[[2]] ; \Omega=$ Parameters $[[3]] ;$
$a=$ Parameters $[[4]] ; \varepsilon=$ Parameters $[[5]] ; t 0=$ Parameters $[[6]]$;
$e p 1=\cos [\omega] \cos [\Omega]-\cos [\iota] \sin [\omega] \sin [\Omega], \cos [\iota] \cos [\Omega] \sin [\omega]+\cos [\omega] \sin [\Omega], \sin [\iota] \sin [\omega] ;$
$e p 2=-\cos [\Omega] \sin [\omega]-\cos [\iota] \cos [\omega] \sin [\Omega], \cos [\iota] \cos [\omega] \cos [\Omega]-\sin [\omega] \sin [\Omega], \cos [\omega] \sin [\iota] ;$
$e p 3=\sin [\iota] \sin [\Omega],-\cos [\Omega] \sin [\iota], \cos [\iota] ;$
$\lambda=\frac{1}{2} \sqrt{a\left(1-\varepsilon^{2}\right)} ;$
$\eta=-\frac{1}{2 a} ;$
$\lambda i=1 / \lambda ;$
$q q=\frac{3}{2} S q r t\left[\left(24 \eta+12 \eta^{2}\right)+\left(3-108 \eta-378 \eta^{2}-324 \eta^{3}-81 \eta^{4}\right) \lambda i^{2}-12 \lambda i^{4}\right] /\left(-3 \lambda i^{2}+1\right)^{3 / 2} \lambda i ;$
$\Psi=-\arcsin [q q] ;$
$U 1=\sqrt{1-3 / \lambda^{2}}+\frac{1 / \lambda^{2}}{1+\sqrt{1-3 / \lambda^{2}}}-\frac{4}{3} \sqrt{1-3 / \lambda^{2}} \sin ^{2}[\Psi / 6] ;$
$U 2=\frac{1}{3}\left(\frac{3 / \lambda^{2}}{1+\sqrt{1-3 / \lambda^{2}}}-\sqrt{3} \sqrt{1-3 / \lambda^{2}} \sin [\Psi / 3]+2 \sqrt{1-3 / \lambda^{2}} \sin ^{2}[\Psi / 6]\right) ;$
$U 3=\frac{1}{3}\left(\frac{3 / \lambda^{2}}{1+\sqrt{1-3 / \lambda^{2}}}+\sqrt{3} \sqrt{1-3 / \lambda^{2}} \sin [\Psi / 3]+2 \sqrt{1-3 / \lambda^{2}} \sin ^{2}[\Psi / 6]\right) ;$
$n a=\frac{1}{\sqrt{U 1-U 3}} ;$
$m a=\frac{U 2-U 3}{U 1-U 3} ;$
$n_{1}=1-\frac{U 2}{U 3}$;
$n 2=\frac{U 2-U 3}{1-U 3} ;$
$\chi=$ JacobiAmplitude $\left[\right.$ EllipticK $\left.[m a]+\frac{\phi}{2 n a}, m a\right] ;$
$u=U 2-(U 2-U 3) \cos [\chi]^{2} ;$
$t F i r s t=\left(1+U 3+\frac{n_{1}^{2}-m a}{2\left(m a-n_{1}\right)\left(n_{1}-1\right)}\right)$ EllipticPi $\left[n_{1}, \chi, m a\right]+\frac{U 3^{2}}{1-U 3}$ EllipticPi $[n 2, \chi, m a] ;$
$t$ Second $=\left(\right.$ EllipticE $[\chi, m a]-\left(1-\frac{m a}{n_{1}}\right)$ EllipticF $\left.[\chi, m a]-\left(n_{1} \sin [2 \chi] \frac{\sqrt{1-m a \sin ^{2}[\chi]}}{\left(2\left(1-n_{1} \sin [\chi]^{2}\right)\right.}\right)\right) ;$
$t=\frac{4 n a}{U 3^{2}} \frac{1+\eta}{\lambda}\left(t\right.$ First $+\frac{n_{1} / 2}{\left(m a-n_{1}\right)\left(n_{1}-1\right)}$ tSecond $) ;$
$\tau$ First $=\frac{U 2-U 3}{\lambda U 2(U 2+(-1+m a) U 3)}$ EllipticE $[\chi, m a]-\frac{1}{\lambda U 2}$ EllipticF $[\chi, m a]$;
$\tau$ Second $=\frac{U 2^{2}+2 m a U 2 U 3+(-1+m a) U 3^{2}}{\lambda U 2 U 3(U 2+(-1+m a) U 3)}$ EllipticPi $\left[1-\frac{U 2}{U 3}, \chi, m a\right] ;$
$\tau$ Third $=\frac{\cos [\chi] \sin [\chi] \sqrt{1-m a \sin [\chi]^{2}}(U 2-U 3)^{2}}{\lambda U 2(U 2+(-1+m a) U 3)\left(\sin [\chi]^{2} U 2+\cos [\chi]^{2} U 3\right)}$;
$\tau=\frac{2 n a}{U 3}(\tau$ First $+\tau$ Second $+\tau$ Third $) ;$
$R 3=2 / u(\cos [\phi] e p 1+\sin [\phi] e p 2) ;$
$V e n=J o i n[R 3, t+t 0, \tau]]$;

Tau $=$ Function $[\{\phi$, Parameters $\}$,
$\iota=$ Parameters $[[1]] ; \omega=$ Parameters $[[2]] ; \Omega=$ Parameters $[[3]] ;$
$a=\operatorname{Parameters}[[4]] ; \varepsilon=$ Parameters $[[5]] ; t 0=$ Parameters $[[6]] ;$
$\lambda=1 / 2 \operatorname{Sqrt}\left[a\left(1-\varepsilon^{2}\right)\right] ; \quad \eta=-\frac{1}{2 a} ; \quad \lambda i=1 / \lambda ;$
$q q=\frac{3}{2} \frac{\sqrt{\left(24 \eta+12 \eta^{2}\right)+\left(3-108 \eta-378 \eta^{2}-324 \eta^{3}-81 \eta^{4}\right) \lambda i^{2}-12 \lambda i^{4}}}{\left(-3 \lambda i^{2}+1\right)^{(3 / 2)}} \lambda i ;$
$\Psi=-\arcsin [q q] ;$
$U 1=\sqrt{1-3 / \lambda^{2}}+\frac{1 / \lambda^{2}}{1+\sqrt{1-3 / \lambda^{2}}}-\frac{4}{3} \sqrt{1-3 / \lambda^{2}} \sin ^{2}[\Psi / 6] ;$
$U 2=\frac{1}{3}\left(\frac{3 / \lambda^{2}}{1+\sqrt{1-3 / \lambda^{2}}}-\sqrt{3} \sqrt{1-3 / \lambda^{2}} \sin [\Psi / 3]+2 \sqrt{1-3 / \lambda^{2}} \sin \Psi / 6^{2}\right) ;$
$U 3=\frac{1}{3}\left(\frac{3 / \lambda^{2}}{1+\sqrt{1-3 / \lambda^{2}}}+\sqrt{3} \sqrt{1-3 / \lambda^{2}} \sin [\Psi / 3]+2 \sqrt{1-3 / \lambda^{2}} \sin [\Psi / 6]^{2}\right) ;$
$n a=\frac{1}{\sqrt{U 1-U 3}} ;$
$m a=\frac{U 2-U 3}{U 1-U 3} ;$
$n_{1}=1-\frac{U 2}{U 3}$;
$n 2=\frac{U 2-U 3}{1-U 3} ;$
$\chi=$ JacobiAmplitude $\left[\right.$ EllipticK $\left.[m a]+\frac{\phi}{2 n a}, m a\right]$;
$\tau$ First $=\frac{U 2-U 3}{\lambda U 2(U 2+(-1+m a) U 3)}$ EllipticE $[\chi, m a]-\frac{1}{\lambda U 2}$ EllipticF $[\chi, m a] ;$
$\tau$ Second $=\frac{U 2^{2}+2 m a U 2 U 3+(-1+m a) U 3^{2}}{\lambda U 2 U 3(U 2+(-1+m a) U 3)}$ EllipticPi $\left[1-\frac{U 2}{U 3}, \chi, m a\right] ;$
$\tau$ Third $=\frac{\cos [\chi] \sin [\chi] \sqrt{1-m a \sin [\chi]^{2}}(U 2-U 3)^{2}}{\lambda U 2(U 2+(-1+m a) U 3)\left(\sin [\chi]^{2} U 2+\cos [\chi]^{2} U 3\right)}$;
$\tau=\frac{2 n a}{U 3}(\tau$ First $+\tau$ Second $+\tau$ Third $\left.)\right] ;$

$$
\text { Funk }[\tau, w 1, w 2, S a t]:=\frac{(w 2-w 1)(\tau-\operatorname{Tau}[w 1, S a t])}{\operatorname{Tau}[w 2, S a t]-\operatorname{Tau}[w 1, S a t]}+w 1 ;
$$

Anomaly $=$ Function $[\{\tau$, Sat $\}$,
$s 1=F u n k[\tau, 0, \pi, S a t] ;$
$s 2=F u n k[\tau, s 1 \times 0.99, s 1 \times 1.01, S a t] ;$
$s 3=F u n k[\tau, s 2 \times 0.99999, s 2 \times 1.00001, S a t]] ;$

$$
\begin{aligned}
& \text { to } \chi \mathbf{F}=\text { Function }\left[\{r i, r f, \theta\}, \quad u_{i}=\frac{2}{M a x[r i, r f]} ; \quad u_{f}=\frac{2}{M i n[r i, r f]} ;\right. \\
& \psi_{i}=3 \arcsin \left[\sqrt{\left.\frac{3}{4}\left(\left(1+\operatorname{Cot}\left[\frac{\theta}{2}\right]^{2}\right)\left(u_{i}-u_{f}\right)^{2}+\left(u_{i}+u_{f}\right)^{2}\left(1+\tan \left[\frac{\theta}{2}\right]^{2}\right)\right)\right] ;}\right. \\
& \mu=\frac{2}{\sqrt{3}} \frac{\tan \left[\frac{\psi_{i}}{3}\right]}{1+\frac{1}{\sqrt{3}} \tan \left[\frac{\psi_{i}}{3}\right]} ; \quad \nu=\frac{2}{\sqrt{1-2 \sin \left[\frac{\psi_{i}}{6}\right]^{2}+\frac{1}{\sqrt{3}} \sin \left[\frac{\psi_{i}}{3}\right]}} ; \\
& \text { Levi } \left.=\text { JacobiCN[ } \frac{\theta}{\nu}, \mu\right] ; \\
& R F i r s t=\sqrt{\left(3 u_{f}-2 \sin \left[\frac{\psi_{i}}{6}\right]^{2}-\sqrt{3} \sin \left[\frac{\psi_{i}}{3}\right]\right)\left(3 u_{i}-2 \sin \left[\frac{\psi_{i}}{6}\right]^{2}-\sqrt{3} \sin \left[\frac{\psi_{i}}{3}\right]\right) ;} \\
& R S e c o n d=\left(( 3 - 3 u _ { f } - 4 \operatorname { s i n } [ \frac { \psi _ { i } } { 6 } ] ^ { 2 } ) ( 3 - 3 u _ { i } - 4 \operatorname { s i n } [ \frac { \psi _ { i } } { 6 } ] ^ { 2 } ) \left(3 u_{f}-2 \sin \left[\frac{\psi_{i}}{6}\right]^{2}+\right.\right. \\
& \left.\left.\quad \sqrt{3} \sin \left[\frac{\psi_{i}}{3}\right]\right)\left(3 u_{i}-2 \sin \left[\frac{\psi_{i}}{6}\right]^{2}+\sqrt{3} \sin \left[\frac{\psi_{i}}{3}\right]\right)\right)^{1 / 2} ; \\
& R D e n=18 \sqrt{3} u_{f} u_{i}-4 \sqrt{3}\left(2+3 u_{f}+3 u_{i}+4 \cos \left[\frac{\psi_{i}}{3}\right]\right) \sin \left[\frac{\psi_{i}}{6}\right]^{2}-12 \sin \left[\frac{\psi_{i}}{3}\right]+ \\
& 18 u_{f} \sin \left[\frac{\psi_{i}}{3}\right]+18 u_{i} \sin \left[\frac{\psi_{i}}{3}\right]-12 \sin \left[\frac{\left(2 \psi_{i}\right)}{3}\right] ; \\
& R 1=-\frac{2 \sqrt{3}\left(\left(3 \cos \left[\frac{\psi_{i}}{3}\right]+\sqrt{3} \sin \left[\frac{\psi_{i}}{3}\right]\right) R F i r s t+R S e c o n d\right)}{R D e n} ; \\
& R 2=-\frac{2 \sqrt{3}\left(-\left(3 \cos \left[\frac{\psi_{i}}{3}\right]+\sqrt{3} \sin \left[\frac{\psi_{i}}{3}\right]\right) R F i r s t+\text { RSecond }\right)}{R D e n} ; \\
& \psi o=I f[A b s[R 1-L e v i]<A b s[R 2-\text { Levi], }
\end{aligned}
$$

FindRoot[
JacobiCN $\left[\frac{1}{2} \theta \sqrt{\cos \left[\frac{x}{3}\right]+\frac{1}{\sqrt{3}} \sin \left[\frac{x}{3}\right]}, \frac{2 \tan \left[\frac{x}{3}\right]}{\frac{1}{\sqrt{3}}+\tan \left[\frac{x}{3}\right]}\right]+$
$\frac{\sqrt{3}}{9 \sqrt{3} u_{f} u_{i}-2 \sqrt{3}\left(2+3 u_{f}+3 u_{i}+4 \cos \left[\frac{x}{3}\right]\right) \sin \left[\frac{x}{6}\right]^{2}-\left(6-9 u_{f}-9 u_{i}\right) \sin \left[\frac{x}{3}\right]-6 \sin \left[\frac{2 x}{3}\right]}$
$\left(\sqrt{\left(\left(1-3 u_{f}+2 \cos \left[\frac{x}{3}\right]\right)\left(1-3 u_{i}+2 \cos \left[\frac{x}{3}\right]\right)\right.}\right.$
$\sqrt{\left(1-3 u_{f}-\cos \left[\frac{x}{3}\right]-\sqrt{3} \sin \left[\frac{x}{3}\right]\right)\left(1-3 u_{i}-\cos \left[\frac{x}{3}\right]-\sqrt{3} \sin \left[\frac{x}{3}\right]\right)}+$

$$
\begin{aligned}
& \left.\left(3 \cos \left[\frac{x}{3}\right]+\sqrt{3} \sin \left[\frac{x}{3}\right]\right) \sqrt{\left.1-3 u_{f}-\cos \left[\frac{x}{3}\right]+\sqrt{3} \sin \left[\frac{x}{3}\right]\right)\left(1-3 u_{i}-\cos \left[\frac{x}{3}\right]+\sqrt{3} \sin \left[\frac{x}{3}\right]\right.}\right) \\
& \left.\left\{x, \psi_{i}\right\}\right]
\end{aligned}
$$

FindRoot[
JacobiCN $\left[\frac{1}{2} \theta \sqrt{\cos \left[\frac{x}{3}\right]+\frac{1}{\sqrt{3}} \sin \left[\frac{x}{3}\right]}, \frac{2 \tan \left[\frac{x}{3}\right]}{\frac{1}{\sqrt{3}}+\tan \left[\frac{x}{3}\right]}\right]+$
$\frac{\sqrt{3}}{9 \sqrt{3} u_{f} u_{i}-2 \sqrt{3}\left(2+3 u_{f}+3 u_{i}+4 \cos \left[\frac{x}{3}\right]\right) \sin \left[\frac{x}{6}\right]^{2}-\left(6-9 u_{f}-9 u_{i}\right) \sin \left[\frac{x}{3}\right]-6 \sin \left[\frac{2 x}{3}\right]}$
$\left(\sqrt{\left(\left(1-3 u_{f}+2 \cos \left[\frac{x}{3}\right]\right)\left(1-3 u_{i}+2 \cos \left[\frac{x}{3}\right]\right)\right.}\right.$
$\sqrt{\left(1-3 u_{f}-\cos \left[\frac{x}{3}\right]-\sqrt{3} \sin \left[\frac{x}{3}\right]\right)\left(1-3 u_{i}-\cos \left[\frac{x}{3}\right]-\sqrt{3} \sin \left[\frac{x}{3}\right]\right)}-$
$\left.\left(3 \cos \left[\frac{x}{3}\right]+\sqrt{3} \sin \left[\frac{x}{3}\right]\right) \sqrt{\left.1-3 u_{f}-\cos \left[\frac{x}{3}\right]+\sqrt{3} \sin \left[\frac{x}{3}\right]\right)\left(1-3 u_{i}-\cos \left[\frac{x}{3}\right]+\sqrt{3} \sin \left[\frac{x}{3}\right]\right.}\right)$,
$\left.\left.\left\{x, \psi_{i}\right\}\right]\right]$;
$\psi=x / . \psi 0 ;$
$u 2=\frac{2}{3} \sin \left[\frac{\psi}{6}\right]^{2}+\sin \left[\frac{\psi}{3}\right] / \sqrt{3} ; \quad u 3=\frac{2}{3} \sin \left[\frac{\psi}{6}\right]^{2}-\frac{1}{\sqrt{3}} \sin \left[\frac{\psi}{3}\right] ;$
$n_{1}=1-\frac{u 2}{u 3} ; \quad n 2=\frac{u 2-u 3}{1-u 3} ; \quad$ 2minusu $3=\frac{2}{\sqrt{3}} \sin \left[\frac{\psi}{3}\right] ;$
$m=\frac{2}{\sqrt{3}} \frac{\tan \left[\frac{\psi}{3}\right]}{1+\frac{1}{\sqrt{3}} \tan \left[\frac{\psi}{3}\right]} ; \quad n=\frac{2}{\sqrt{1-2 \sin \left[\frac{\psi}{6}\right]^{2}+\frac{1}{\sqrt{3}} \sin \left[\frac{\psi}{3}\right]}} ;$
$\chi 1=\arccos \left[\sqrt{\left(u 2-u_{i}\right) / u 2 \text { minusu } 3}\right] ;$
$\Delta \lambda p=\operatorname{Re}[n($ EllipticK $[m]-$ EllipticF $[\chi 1, m])] ;$
$\chi 2=I f\left[\Delta \lambda p>\theta, \arccos \left[\sqrt{\frac{u 2-u_{f}}{u 2 \text { minusu } 3}}\right], \arccos \left[-\sqrt{\left(\left(u 2-u_{f}\right) / u 2 \text { minusu } 3\right)}\right] ;\right.$
$o u t=\chi 1, \chi 2 ;$
TimeOf Flight $=\frac{2 n}{u 3^{2}} \frac{2}{3 \sqrt{3}} \sin \left[\frac{\psi}{2}\right]\left(\left(1+u 3+\frac{\left(n_{1}^{2}-m\right)}{2\left(m-n_{1}\right)\left(n_{1}-1\right)}\right)\right.$
If $\left[\operatorname{Re}\left[n_{1}\right]<1\right.$, EllipticPi $\left[n_{1}, \chi 2, m\right], 2 Q u o t i e n t[\chi 2, \pi]$ EllipticPi $\left[n_{1}, \frac{\pi}{2}, m\right]-$
$\frac{1}{\left(n_{1}-1\right) \sqrt{1-m}}$ EllipticPi $\left.\left[\frac{n_{1}}{n_{1}-1}, \chi 2-\frac{\pi}{2}, \frac{m}{m-1}\right]\right]+\frac{u 3^{2}}{1-u 3}$
If $\left[\operatorname{Re}\left[n_{1}\right]<1\right.$, EllipticPi $\left[n_{1}, \chi 2, m\right], 2 Q u o t i e n t[\chi 2, \pi]$ EllipticPi $\left[n_{1}, \frac{\pi}{2}, m\right]-$
$\frac{1}{\left(n_{1}-1\right) \sqrt{1-m}}$ EllipticPi $\left.\left[\frac{n_{1}}{n_{1}-1}, \chi 2-\frac{\pi}{2}, \frac{m}{m-1}\right]\right]+$
$\frac{n_{1} / 2}{\left(m-n_{1}\right)\left(n_{1}-1\right)}($ EllipticE $[\chi 2, m]-$
$\left(1-\frac{m}{n_{1}}\right)$ EllipticF $\left.\left.[\chi 2, m]-\frac{n_{1} \sin [2 \chi 2] \sqrt{1-m \sin [\chi 2]^{2}}}{2\left(1-n_{1} \sin [\chi 2]^{2}\right)}\right)\right)-$
$\frac{2 n}{u 3^{2}} \frac{2}{3 \sqrt{3}} \sin \left[\frac{\psi}{2}\right]\left(\left(1+u 3+\frac{n_{1}^{2}-m}{2\left(m-n_{1}\right)\left(n_{1}-1\right)}\right)\right.$
If $\left[\operatorname{Re}\left[n_{1}\right]<1\right.$, EllipticPi $\left[n_{1}, \chi 1, m\right], 2 Q u o t i e n t[\chi 1, \pi]$ EllipticPi $\left[n_{1}, \frac{\pi}{2}, m\right]-$
$\frac{1}{\left(n_{1}-1\right) \sqrt{1-m}}$ EllipticPi $\left.\left[\frac{n_{1}}{n_{1}-1}, \chi 1-\frac{\pi}{2}, \frac{m}{m-1}\right]\right]+\frac{u 3^{2}}{1-u 3}$
$\operatorname{If}\left[\operatorname{Re}\left[n_{1}\right]<1\right.$, EllipticPi $\left[n_{1}, \chi 1, m\right], 2 Q u o t i e n t[\chi 1, \pi]$ EllipticPi $\left[n_{1}, \frac{\pi}{2}, m\right]$
$-\frac{1}{\left(n_{1}-1\right) \sqrt{1-m}}$ EllipticPi $\left.\left[\frac{n_{1}}{n_{1}-1}, \chi 1-\frac{\pi}{2}, \frac{m}{m-1}\right]\right]+$
$\frac{n_{1} / 2}{\left(m-n_{1}\right)\left(n_{1}-1\right)}($ EllipticE $[\chi 1, m]$
$-\left(1-\frac{m}{n_{1}}\right)$ EllipticF $\left.\left.\left.[\chi 1, m]-\frac{n_{1} \sin [2 \chi 1] \sqrt{1-m \sin [\chi 1]^{2}}}{2\left(1-n_{1} \sin [\chi 1]^{2}\right)}\right)\right)\right]$;

$$
\begin{aligned}
& \text { Fun }=\text { Function }[\{\text { Casi }, \text {, Tocke }\}, \\
& \operatorname{Prvi}=\operatorname{Premica} 1[\text { Tocke }[[1]], \text { Casi }[[1]]-t \text {, Tocke }[[2]], \text { Casi } i[2]]-t \text {, } \\
& \operatorname{Tocke}[[3]], \operatorname{Casi}[[3]]-t, \operatorname{Tocke}[[4]], \operatorname{Casi}[[4]]-t] \text {; } \\
& \operatorname{Drugi}=\operatorname{Premica1}[\text { Tocke }[[2]], \operatorname{Casi}[[2]]-t, \text { Tocke }[[3]], \text { Casi }[[3]]-t, \\
& \text { Tocke }[11]], \text { Casi }[[1]]-t \text {, Tocke }[44]], \operatorname{Casi}[[4]]-t] ; \\
& A=\operatorname{Prvi}[[1]] ; \quad B=\operatorname{Drugi}[[1]] ; \\
& s 1=\operatorname{Prvi}[[2]] ; \quad s 2=\operatorname{Drugi}[[2]] ; \\
& d e n=\operatorname{Cross}[s 1, s 2] \text {; } \\
& u=\frac{(A-B) \cdot s 2 \text { s1.s2-(A-B).s1 }}{\text { den.den }} ; \\
& v=\frac{(A-B) \cdot s 2-(A-B) \cdot s 1 s 1 . s 2}{\text { den.den }} ; \\
& r e z=\{A+u s 1, B+v s 2\} ; \\
& \operatorname{Rad}=\operatorname{Table}[\sqrt{(\text { Tocke }[[i]]-\operatorname{rez}[[1]]) .(\text { Tocke }[[i]]-\operatorname{rez}[[1]])},\{i, 4\}]] ;
\end{aligned}
$$

Iteriraj $[\{s s 1, s s 2\}]=$ Function $[\{$ Casi, Tocke $\}$,
$Y 1=F u n[C a s i, s s 1, T o c k e]-C a s i+\{s s 1, s s 1, s s 1, s s 1\} ;$
$Y 2=F u n[C a s i, s s 2, T o c k e]-C a s i+\{s s 2, s s 2, s s 2, s s 2\} ;$
$s s 0=$ Table $\left[\frac{Y 2[[i]] s s 1-Y 1[[i]] s s 2}{Y 2[[i]]-Y 1[[i]]},\{i, 4\}\right] ;$
$\operatorname{Avg}=\operatorname{Sum}[s s 0[[i]],\{i, 4\}] / 4 ; \operatorname{Sigma}=\operatorname{Sqrt}\left[\operatorname{Sum}\left[(s s 0[[i]]-\operatorname{Avg})^{2},\{i, 4\}\right]\right] / 4 ;$
$M e j e=I f\left[\right.$ Sigma $>0$, Sigma, 10 $\left.{ }^{-} 12\right] ;\{$ Avg - Meje, Avg + Meje $\left.\}\right]$;
$\qquad$

## Subprocedures

- Odseka: $a, b, c$ are the sides of a triangle (with vertices A, B, C), let $c$ be considered a base, so that the vertices at its ends have coordinates $A=\{0,0\}$ and $B=\{0, c\}$; the routine returns $\{\chi, \eta, h\}$ such that the coordinates of vertex C are $\mathrm{C}=\{c \chi, h\}$
- Premica1: let $r r 1 \ldots r r 4$ be the coordinates of 4 points and $d d 1 d o t s d d 4$ be the radii of four spheres $(S 1, \ldots, S 4)$ centred at $r r 1 \ldots r r 4$. The intersection of S 1 and S 2 defines a plane P12, and the intersection of S3 and S4 defines a plane P23. The intersection of P12 and P23 is a straight line with equation:

$$
\begin{equation*}
\vec{r}(s)=\vec{a}+\hat{n} s \tag{B.1}
\end{equation*}
$$

the output of Premica1 are the two 3 - vectors $\vec{a}$ and $\hat{n}(\hat{n}$ is a unit vector)

- Fun $\left[\right.$ Casi, $t_{r}$, Tocke $]$ : Tocke is the array of position vectors of the 4 satellites at the moment of emission of their signals and Casi is the array $t_{i}-t_{0}(i=1 \ldots 4)$, where $t_{i}$ are the global times (Schwarzschild, not proper) at the moment of emission of their signals and $t_{0}$ is the current approximation to the time of the observer. The variable rez holds two position vectors to the point on line 1 and line 2 where the two lines meet at closest distance (see text above); their average position are the best coordinates for the position of the Galileo user if his time is $t_{0}-t_{r}$. The output array $\mathbf{R a d}$ gives the 4 times of flight (here in flat space) to the satellites calculated from the assumed position in time; Rad is the explicit output of this routine and rez is the variable set by this routine (see Iterate).

$$
\begin{aligned}
& \text { Matrix }=\text { Function }[\{\text { Tocke, Pos }\} \\
& \text { DirVek }=\text { Table }\left[\frac{(\text { Pos }- \text { Tocke }[[i]])}{\sqrt{(\text { Pos }- \text { Tocke }[[i]]) \cdot(\text { Pos }- \text { Tocke }[[i]])}},\{i, 4\}\right]
\end{aligned}
$$

```
\(\mathbf{R H S}=\) Function \([\{\) Casi, Tocke, Pos \(\}\),
Des \(=\) Table \([\) Casi \([[i]]-\)
to \(\left.\left.\left.F\left[\sqrt{\text { Tocke }[[i]] . \text { Tocke }[[i]]}, \sqrt{\text { Pos.Pos }}, \frac{\arccos [\text { Pos.Tocke }[[i]]}{\sqrt{\text { Tocke }[[i]] \cdot \text { Tocke }[[i]]} \sqrt{\text { Pos.Pos }}}\right]\right],\{i, 4\}\right]\right] ;\)
```

$\qquad$
$\operatorname{Cas}[A]:=A[[4]] ;$
Polozaj $[A]:=$ Table $[A[[i]],\{i, 3\}] ;$

Premica1 $=$ Function $[\{r r 1, d d 1, r r 2, d d 2, r r 3, d d 3, r r 4, d d 4\}$,
$c=\sqrt{(r r 2-r r 1) \cdot(r r 2-r r 1)} ; \quad n 11=\operatorname{Cross}[r r 2, r r 1] ;$
$n 1=\frac{n 11}{\sqrt{n 11 . n 11}} ; \quad n 2=\operatorname{Cross}[(r r 2-r r 1) / c, n 1] ;$
Prm $=$ Odseka $[d d 1, d d 2, c] ; \quad c 1=\sqrt{(r r 4-r r 3) \cdot(r r 4-r r 3)} ;$
$n 11 p=\operatorname{Cross}[r r 4, r r 3] ; \quad n 1 p=\frac{n 11 p}{\sqrt{n 11 p . n 11 p}}$;
$n 2 p=\operatorname{Cross}[(r r 4-r r 3) / c 1, n 1 p] ;$
Prmp $=$ Odseka $[d d 3, d d 4, c 1]$;
$\nu_{1}=\operatorname{Cross}[n 1, n 2] ;$
$\nu_{2}=\operatorname{Cross}[n 1 p, n 2 p] ;$
$a=r r 1+\operatorname{Prm}[[1]](r r 2-r r 1) ;$
$b=r r 3+\operatorname{Prmp}[[1]](r r 4-r r 3)$;
prem $=\operatorname{Cross}\left[\nu_{1}, \nu_{2}\right]$;
Nprem $=$ prem.prem;
$R=\left\{\frac{a . \nu_{1}}{\text { Nprem }} \operatorname{Cross}\left[\nu_{2}\right.\right.$, prem $]+\frac{\text { b. } \nu_{2}}{\text { Nprem }} \operatorname{Cross}\left[\right.$ prem,$\left.\left.\left.\nu_{1}\right], \frac{\text { prem }}{\sqrt{\text { Nprem }}}\right\}\right] ;$

$$
\begin{aligned}
& \text { eps }=10^{-8} ; \quad \text { to test if one of angles }>\pi / 2 \\
& \text { Odseka }=\text { Function }[\{a, b, c\}, \\
& \alpha=\frac{a}{c} ; \beta=\frac{b}{c} ; \sigma=\frac{1}{2}(\alpha+\beta+1) ; \\
& A 2=\sigma(\sigma-\alpha)(\sigma-\beta)(\sigma-1) ; \\
& h=2 \sqrt{A 2} ; \quad \chi=\sqrt{\alpha^{2}-4 A 2} ; \quad \eta=\sqrt{\beta^{2}-4 A 2} ; \\
& \text { ven }=I f[\text { Abs }[\chi+\eta-1]<e p s,\{\chi, \eta, h\}, \operatorname{If}[A b s[-\chi+\eta-1]<e p s,\{-\chi, \eta, h\}, \\
& \operatorname{If}[\text { Abs }[\chi-\eta-1]<e p s,\{\chi,-\eta, h\},\{ \}]]]] ; \\
& \quad \text { END ODSEKA - }
\end{aligned}
$$

## ElLIPTIC INTEGRALS AND FUNCTIONS

In this report, we adopted the following definitions for elliptic integrals and functions (Wolfram 1996).

Elliptic integral of the first kind

$$
\begin{equation*}
\mathrm{F}(\phi \mid m)=\int_{0}^{\phi} \frac{\mathrm{d} u}{\sqrt{1-m \sin ^{2} u}} \tag{C.1}
\end{equation*}
$$

## Complete elliptic integral of the first kind

$$
\begin{equation*}
\mathrm{K}(m)=\mathrm{F}\left(\left.\frac{\pi}{2} \right\rvert\, m\right)=\int_{0}^{\pi / 2} \frac{\mathrm{~d} u}{\sqrt{1-m \sin ^{2} u}} \tag{C.2}
\end{equation*}
$$

Elliptic integral of the second kind

$$
\begin{equation*}
\mathrm{E}(\phi \mid m)=\int_{0}^{\phi} \sqrt{1-m \sin ^{2} u} \mathrm{~d} u \tag{C.3}
\end{equation*}
$$

Elliptic integral of the third kind

$$
\begin{equation*}
\Pi(n ; \phi \mid m)=\int_{0}^{\phi} \frac{\mathrm{d} u}{\left(1-n \sin ^{2} u\right) \sqrt{1-m \sin ^{2} u}} \tag{C.4}
\end{equation*}
$$

If $n>1$ and $\phi>\arcsin \frac{1}{\sqrt{n}}$ then the value of $\Pi$ is complex. To get the real values, this integral has to be evaluated at a different branch:

$$
\begin{equation*}
\Pi(n ; \phi \mid m) \rightarrow \frac{-1}{(n-1) \sqrt{1-m}} \Pi\left(\frac{n}{n-1} ; \left.\phi-\frac{\pi}{2} \right\rvert\, \frac{m}{m-1}\right) \tag{C.5}
\end{equation*}
$$

Jacobi amplitude

$$
\begin{equation*}
\text { If } u=\mathrm{F}(\phi \mid m) \text { then } \phi=\operatorname{am}(u \mid m) . \tag{C.6}
\end{equation*}
$$

## Jacobi elliptic functions

If $\phi=\operatorname{am}(u \mid m)$ then:

$$
\begin{align*}
\operatorname{sn}(u \mid m) & =\sin \phi  \tag{C.7}\\
\operatorname{cn}(u \mid m) & =\cos \phi  \tag{C.8}\\
\operatorname{dn}(u \mid m) & =\sqrt{1-m \sin ^{2} \phi} \tag{C.9}
\end{align*}
$$

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[^0]:    *This is a delicate problem because for this one has to define a global reference frame in which the orbital parameters of the constellation of satellites are known. We will introduce this complication later.

[^1]:    *The Earth's local inertial coordinate frame is precessing with respect to the global inertial frame tied to distant stars. The precessions are due to different gravitational perturbations of Earth's multipole moments, gravitational perturbations of the Moon, Sun and planets. These motions can be well modeled, but are not the matter of this report.
    ${ }^{\dagger}$ To recover usual units one can replace $M=G m_{\oplus} / c^{2}$ when measuring distance and $M=G m_{\oplus} / c^{3}$ when measuring time, where $m_{\oplus}$ is the mass of the Earth in the usual units.

[^2]:    ${ }^{*}$ Note that $\sin \theta=+\sqrt{1-\cos ^{2} \theta}$.

[^3]:    *The problem of the atmospheric perturbations is not in the scope of this paper.

[^4]:    ${ }^{*}$ This is certainly not a realistic assumption with respect to each single event, but may not be so bad on long term average.

[^5]:    ${ }^{*}$ Here it is more natural to use the orbital energy and the areal velocity than the parameters $A$ and $B$ introduced in Sec. 2.

[^6]:    ${ }^{*}$ Keeping in mind that the value of $M$ for Earth is only few millimeters, and that the radius of satellites' orbits is about 30.000 km , the need for type $\boldsymbol{B}$ orbits could arise only if e.g. two satellites are almost perfectly one below another, or the receiver on Earth is almost perfectly below a satellite.

