Relativistic Positioning Systems: Numerical Simulations

Workshop – Relativistic Positioning Systems and their Scientific Applications

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SATELLITE MOTIONS I

GPS AND GALILEO SATELLITE CONSTELLATIONS ARE SIMULATED.

- Satellite trajectories are assumed to be circumferences in the Schwarzschild space-time created by an ideal spherically symmetric Earth.
- The angular velocity is $\Omega = (GM_{\oplus}/R^3)^{1/2}$
- Coordinate and proper times are related as follows: $\gamma = \frac{dt}{d\tau} = \left(1 - \frac{3GM_{\oplus}}{R}\right)^{-1/2}$.
- angles θ and ϕ fixes the orbital plane (see arXiv:1112.6054[gr-qc] $\equiv$ Astrophys. Space Sci., 337, 1-issue, 10-01-2012 $\equiv$ paper I for details), and the angle $\alpha_A(\tau) = \alpha_{A0} - \Omega\gamma\tau$ localizes the satellite on its trajectory

This simple model is good enough as a background configuration. Deviations with respect to the background satellite world lines will be necessary to develop our study about positioning accuracy (see below).

Other known world lines (no circumferences) of Schwarzschild space-time might be easily implemented in the code, but the new background satellite configurations would lead to qualitatively comparable numerical results; at least, for the problems considered here

SATELLITE MOTIONS II

SUMMARY (see paper I for details)

- A given orbital plane is characterized by the constant angles θ and ϕ
- The angle $\alpha_A(\tau)$ may be calculated for every τ
- From these three angles and the proper time τ, the satellite inertial coordinates (x^1, x^2, x^3, x^4) may be easily found. This means that the world lines of the background satellites [functions $y^\alpha = x^\alpha_A(\tau^A)$] are known for every satellite A
- HENCE, given the emission coordinates $(\tau^1, \tau^2, \tau^3, \tau^4)$ of a receiver (user), the inertial coordinates of the four satellites –at emission times– are known. This is necessary for positioning

FROM INERTIAL TO EMISSION COORDINATES

FROM $x^{\alpha} \equiv (x, y, z, t)$ TO $\tau^{A} \equiv (\tau^1, \tau^2, \tau^3, \tau^4)$

It is assumed that photons move in the Minkowski space-time, whose metric has the covariant components $\eta_{\alpha\beta}$. This approach is good enough for us

Since photons follow null geodesics from emission to reception, the following algebraic equations must be satisfied:

$$\eta_{\alpha\beta}[x^{\alpha} - x^{\alpha}_{A}(\tau^{A})][x^{\beta} - x^{\beta}_{A}(\tau^{A})] = 0 \; . \quad (1)$$

These four equations must be NUMERICALLY solved to get the four emission coordinates τ^{A}, where index A numerates the satellites.

The four proper times are the unknowns in the system (1), which may be easily solved by using the well known Newton-Raphson method. A code has been designed to implement this method. It uses multiple precission. Appropriate tests have been performed

Since the satellite world lines are known, functions $x^{\alpha}_{A}(\tau^{A})$ may be calculated for any set of proper times $\tau^1, \tau^2, \tau^3, \tau^4$, thus, the left hand side of Eqs. (1) can be computed and, consequently, the Newton-Raphson method may be applied

FROM EMISSION TO INERTIAL COORDINATES

FROM $\tau^A \equiv (\tau^1, \tau^2, \tau^3, \tau^4)$ TO $x^\alpha \equiv (x, y, z, t)$

Given four emission cordinates τ^A, Eqs. (1) could be numerically solved to get the unknowns x^α, that is to say, the inertial coordinates; however, this numerical method is not used. It is better the use of an analytical formula giving x^α in terms of τ^A, which is due to B. Coll, J.J. Ferrando, & J.A. Morales-Lladosa (Class. Quantum Grav., 27, 2010, 065013)

The analytical formula is preferable because of the following reasons:

- The numerical method based on Eqs. (1) is more time consuming
- The analytical formulation of the problem allows us a systematic and clear discussion of the bifurcation problem, and also a study of the positioning errors close to situations of vanishing Jacobian

The analytical formula has been presented by J. A. Morales-Lladosa in the previous talk. In particular, this formula involves function χ^2 and the discriminant Δ, which may be calculated from $(\tau^1, \tau^2, \tau^3, \tau^4)$.

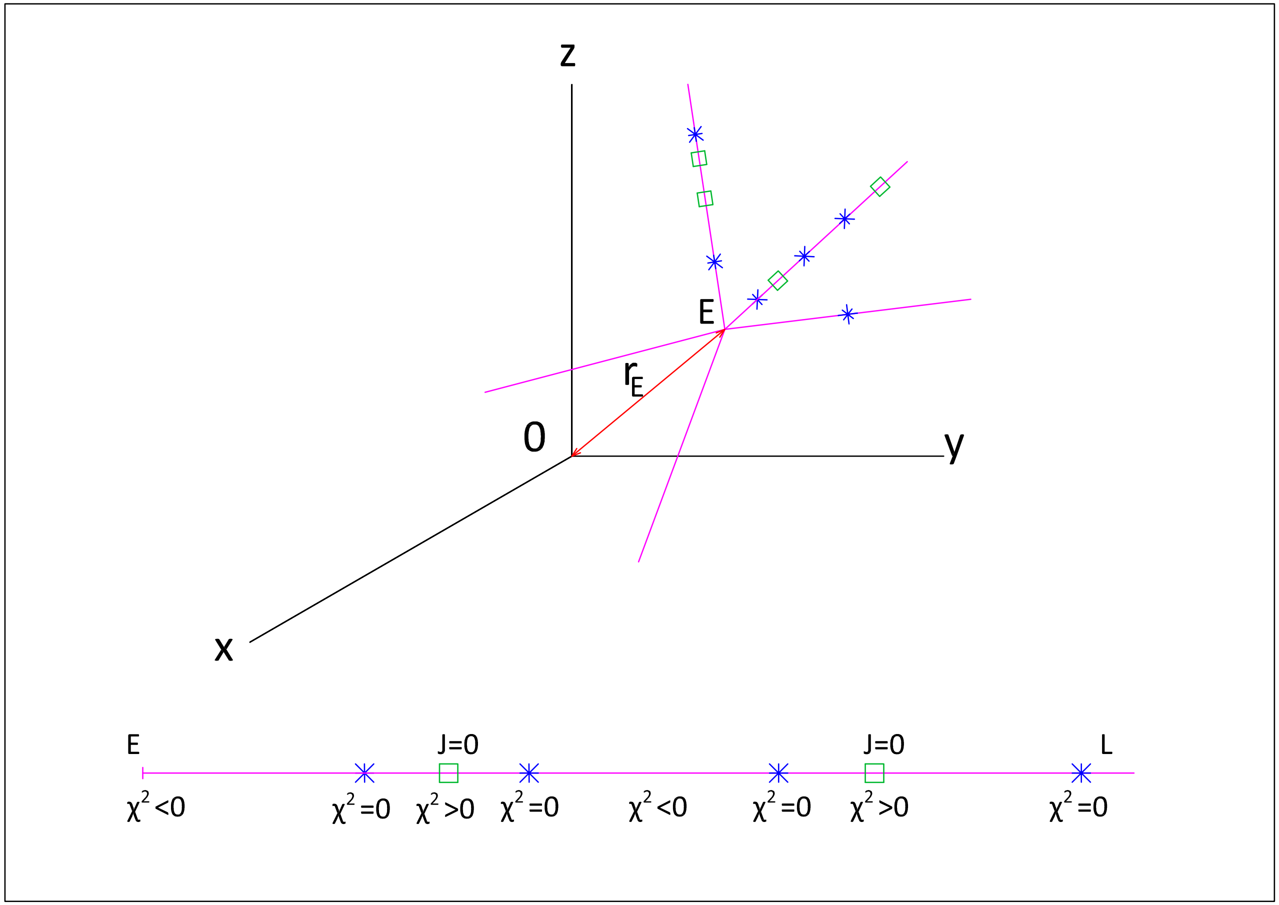
SOME PREVIOUS THEORETICAL RESULTS

By using: the analytical formula giving the inertial coordinates in terms of the emission ones, and some basic relations of Minkowski space-time, the following conclusions have been previously obtained (summary of the previous talk)

- for $\chi^2 \leq 0$, there is only a positioning (past-like) solution
- for $\chi^2 > 0$ there are two positioning solutions; namely, there are two sets of inertial coordinates (two physical real receivers) associated to the same emission coordinates $(\tau^1, \tau^2, \tau^3, \tau^4)$
- the Jacobian J of the transformation giving the emission coordinates in terms of the inertial ones vanishes if and only if the discriminant Δ vanishes
- the Jacobian J may only vanish if $\chi^2 > 0$; namely, in the region of double positioning (bifurcation)
- The Jacobian J may only vanish if the lines of sight –at emision times– of the four satellites belong to the same cone)

These conclusions are basic for the numerical estimates and discussions presented in next slides. In particular the third item is used to get the points of vanishing Jacobian. Close to these points positioning errors are very large

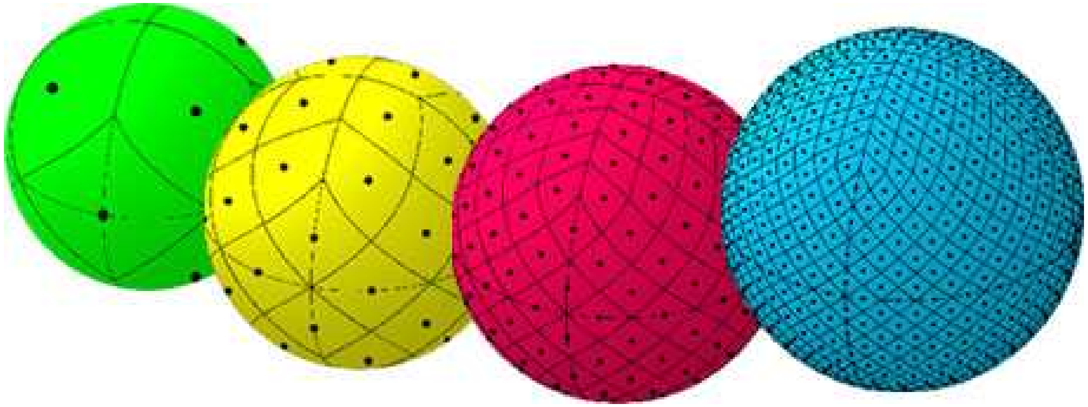
REGION STRUCTURE ANALYSIS



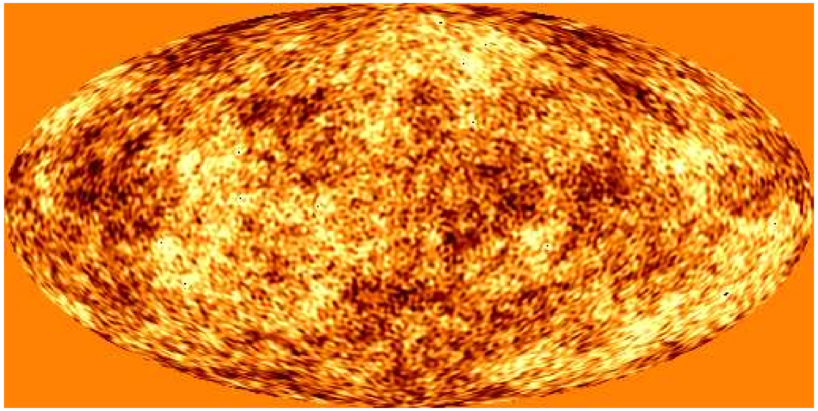
3D sections ($t = constant$) of the so-called *region* are considered. Point E is an arbitrary center. Its distance to the origin O is the Earth radius. 3072 directions starting from E cover the 3D sections. Along each direction our study is restricted to $0 < L < L_{max} = 10^5\ Km$. We look for the zeros of χ^2 and J. The Jacobian only may vanish in the segments where $\chi^2 > 0$, which are limited by the first and second or by the third and fourth χ^2-zeros

REPRESENTATION TECHNIQUES

HEALPIX PIXELISATION(Hierarchical, Equal Area, and iso-Latitude PIXelisation)

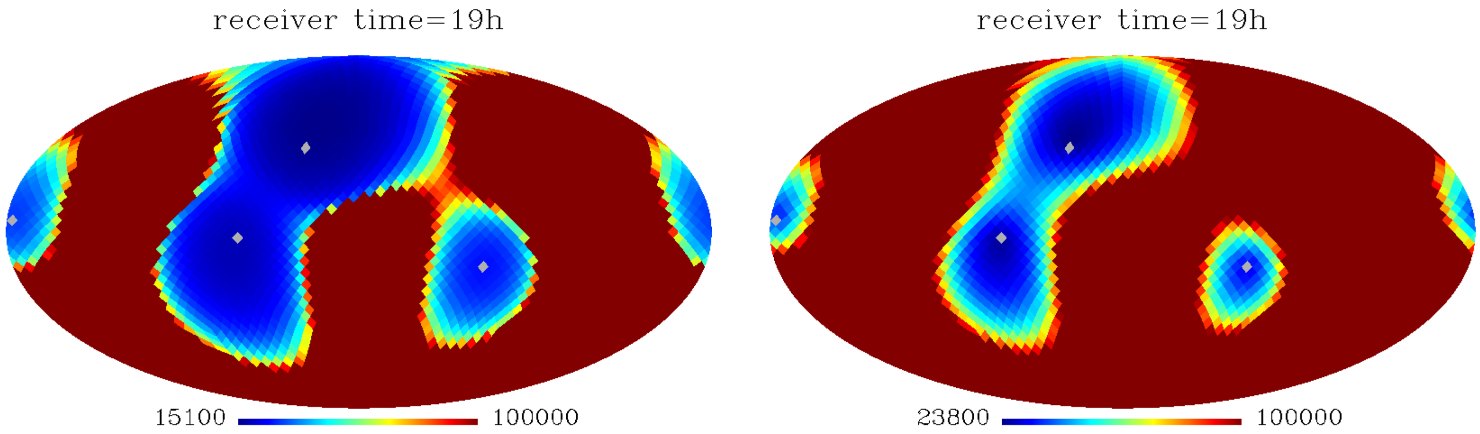


MOLLWEIDE PROJECTION



We use 3072 (12288) pixels in the region (co-region) studies, hence, the pixel angular area is 64 (16) times the mean angular area of the full moon

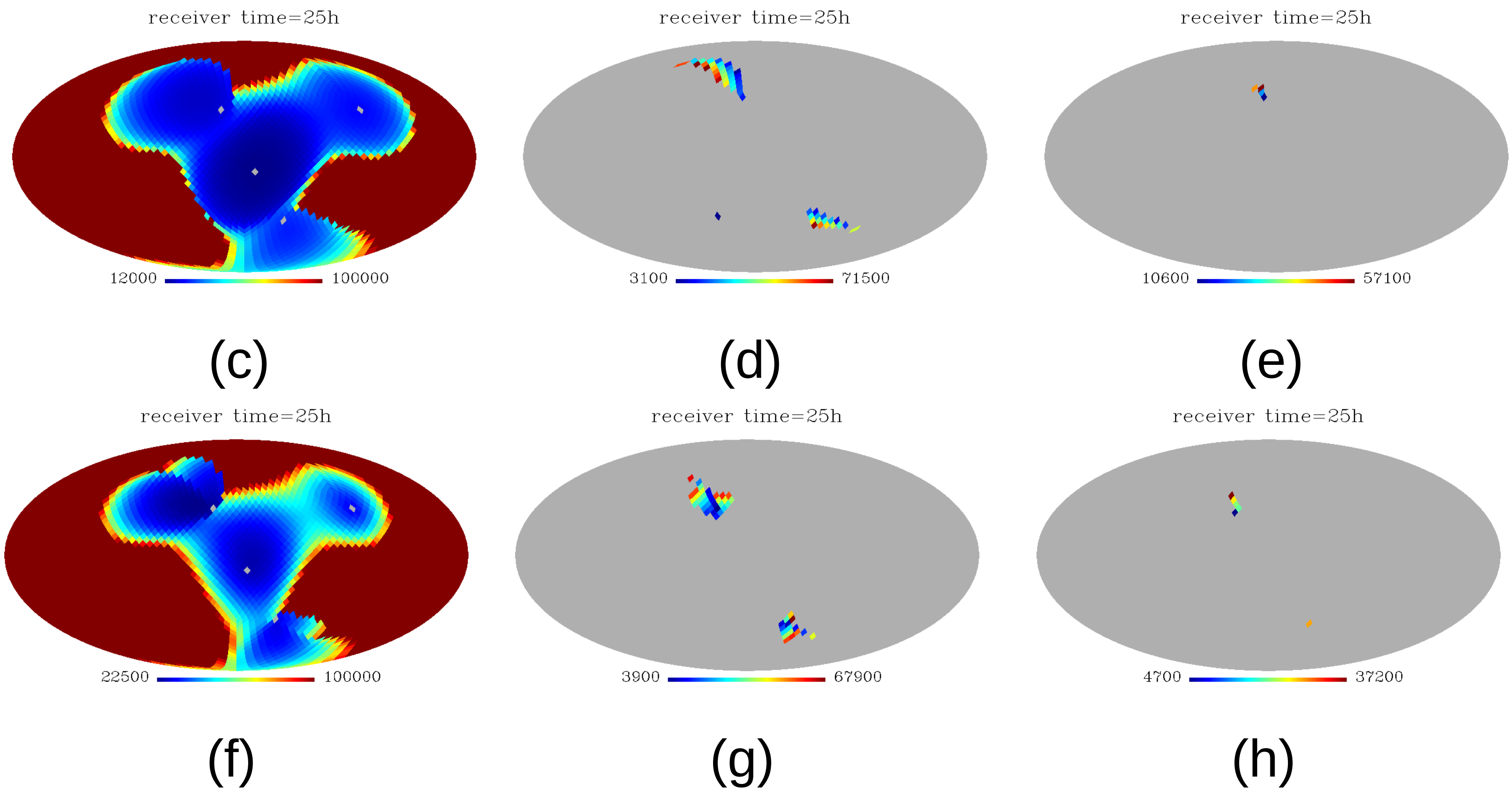
REGION RESULTS I



(a) (b)

- section $x^4 = t = 19\ h$ for $L < 10^5\ Km$ (paper I)
- white pixels indicate satellite positions at emission times. Function χ^2 in (a) [J in (b)] does not vanish for $L < 10^5\ Km$ along red pixels directions
- (a): surface $\chi^2 = 0$. For this choice of satellites, χ^2 only vanishes one time for each direction at a distance from E, $L^\star$. Color bar measures $L^\star$.
- (b): surface $J = 0$. The Jacobian only vanishes one time for each direction at a distance from E $L > L^\star$. Color bar measures $L - L^\star$.
- There are cases in which χ^2 and J vanishes various times

REGION RESULTS II

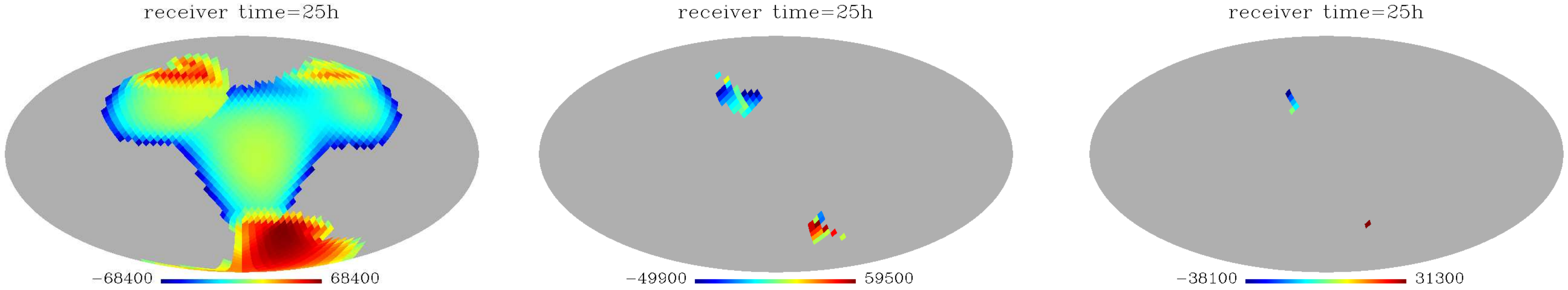


- section $x^4 = t = 25\ h$ for $L < 10^5\ Km$
- (c), (d) and (e) [(f), (g) and (h)] panels represent the first second and third zeros of function $\chi^2 = 0$ [$J = 0$].

REGION RESULTS II

- meaning of the color bars
  - (c) distance from E to the first zero of χ^2
  - (d) distance from the first to the second zero of χ^2
  - (e) distance from the second to the third zero of χ^2
  - (f) distance from E to the first zero of J
  - (g) distance from the first to the second zero of J
  - (h) distance from the second to the third zero of J
- For the chosen satellites and $x^4 = t = 25\ h$, there are no more χ^2 and J zeros for $L < L_{max} = 10^5\ Km$
- there exist a second zero of χ^2 and J only for a few directions, and a third zero of χ^2 or J is very rare
- In what a segment are located the zeros of J? In the first one (between the first and second zeros of χ^2) or in the second one? Let us see next slide to answer this question.

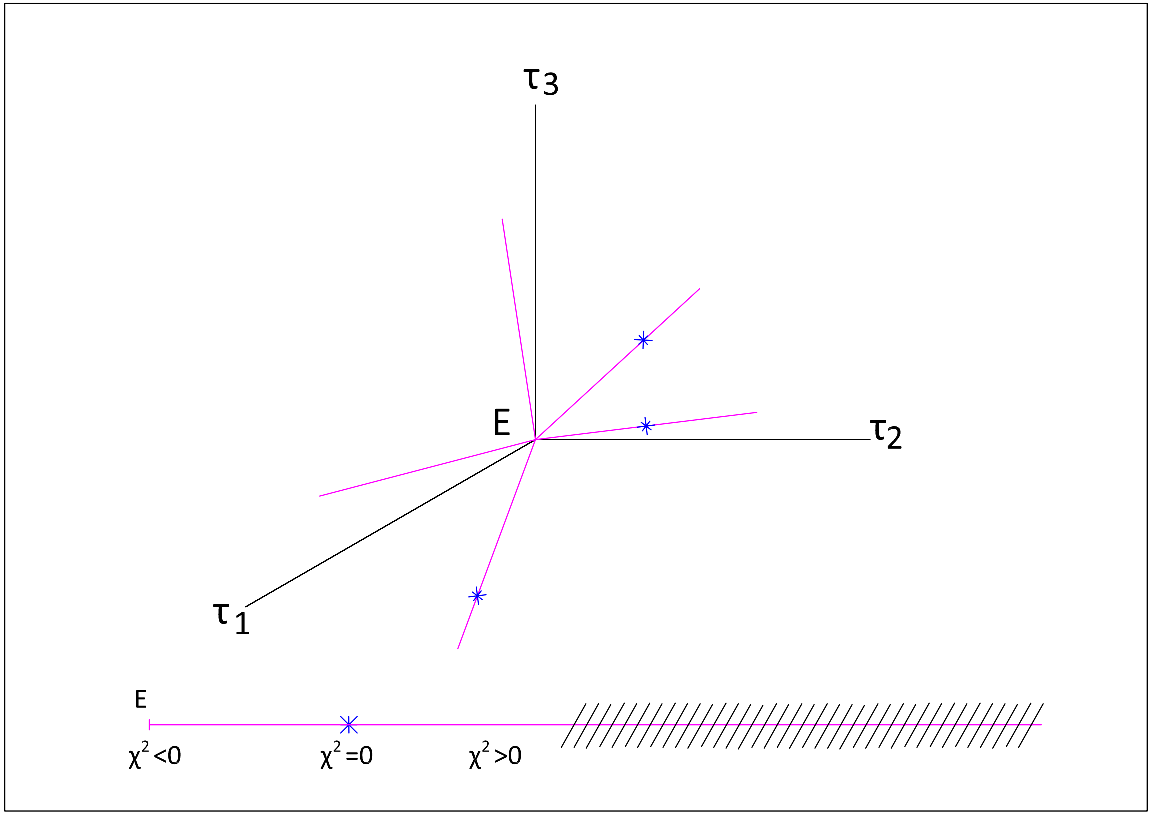
REGION RESULTS III



(i) (j) (k)

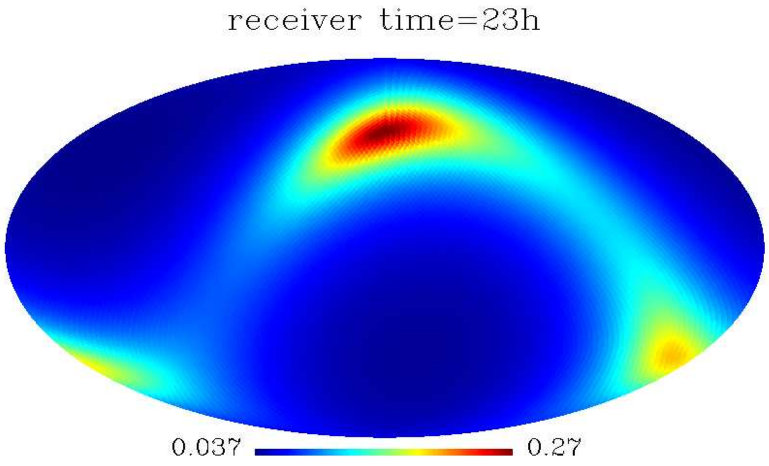
- (i) color bar represents the distance from E to the second zero of χ^2 minus the distance from E to the first J-zero
- (j) color bar represents the distance from E to the second zero of χ^2 minus the distance from E to the second J-zero
- (k) color bar represents the distance from E to the second zero of χ^2 minus the distance from E to the third J-zero
- In every panel, positive values indicate that the corresponding zero of J belongs to the first segment (between the first and second zeros of χ^2)
- In every panel, negative values indicate that the corresponding zero of J belongs to the second segment (beyond the third zero of χ^2)

CO-REGION STRUCTURE ANALYSIS

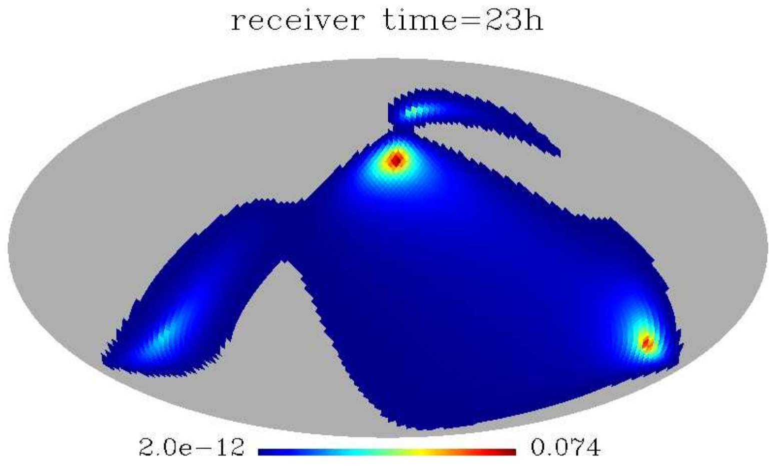


3D sections ($\tau^4 = constant$) of the *co-region* are considered. The emission coordinates of point E are found from the inertial ones of the region center (see above). 12288 directions starting from E cover every 3D section. We look for the zeros of χ^2. For any direction only one zero is found at a time distance λ_- from E. From λ_- to a certain λ_{max}, there are bifurcation with positioning (past-like) or non-positioning (future-like) character. For $\lambda > \lambda_{max}$ we are outside the co-region (the emission-reception conditions are not satisfied)

CO-REGION RESULTS



(l)



(m)

- (l) colors measure the time distance from E to the point where χ^2 vanishes
  - (m) for the grey pixels (directions), there are no positioning solutions (but future-like solutions) between the point where χ^2 vanishes and the first point where the emission-reception conditions are not satisfied
  - (m) if there are two positioning solutions between the point where χ^2 vanishes and the first point where the emission-reception conditions are not satisfied, bar colors measure the distance between these two points. In the interval limited by these points there are bifurcation ($\chi^2 > 0$)

POSITIONING ERRORS AND SATELLITE UNCERTAINTIES I

(a): The background world lines of the satellites are known. Their equations are $y^\alpha = x^\alpha_A(\tau^A)$.

(b): Given the inertial coordinate x^α of a detection event, the above world lines, Eqs. (1), the Newton-Raphson method, and multiple presission may be used to find the emission coordinates $\tau^1, \tau^2, \tau^3, \tau^4$ with very high accuracy.

(c): Finally, the chosen inertial coordinates x^α may be recovered from the emission ones –with very high accuracy– by using the analytical solution found by Coll, Ferrando, & Morales-Lladosa (previous talk).

($a^\star$) Let us now suppose that there are uncertainties in the satellite world lines, whose equations are $y^\alpha = x^\alpha_A(\tau^A) + \xi^\alpha_A$, where ξ^α_A are deviations with respect to the background world lines due to known or unknown external actions on the satellites.

($b^\star$): Given the same inertial coordinate x^α as in (b), the equations $\eta_{\alpha\beta}[x^\alpha - x^\alpha_A(\tau^A) - \xi^\alpha_A][x^\beta - x^\beta_A(\tau^A) - \xi^\beta_A] = 0$, the Newton-Raphson method, and multiple presission may be used to get the perturbed emission coordinates $\tau^1 + \Delta(\tau^1), \tau^2 + \Delta(\tau^2), \tau^3 + \Delta(\tau^3), \tau^4 + \Delta(\tau^4)$. Since the time deviations $\Delta(\tau^A)$ are all small, quantities ξ^α_A may be assumed to be constant.

POSITIONING ERRORS AND SATELLITE UNCERTAINTIES II

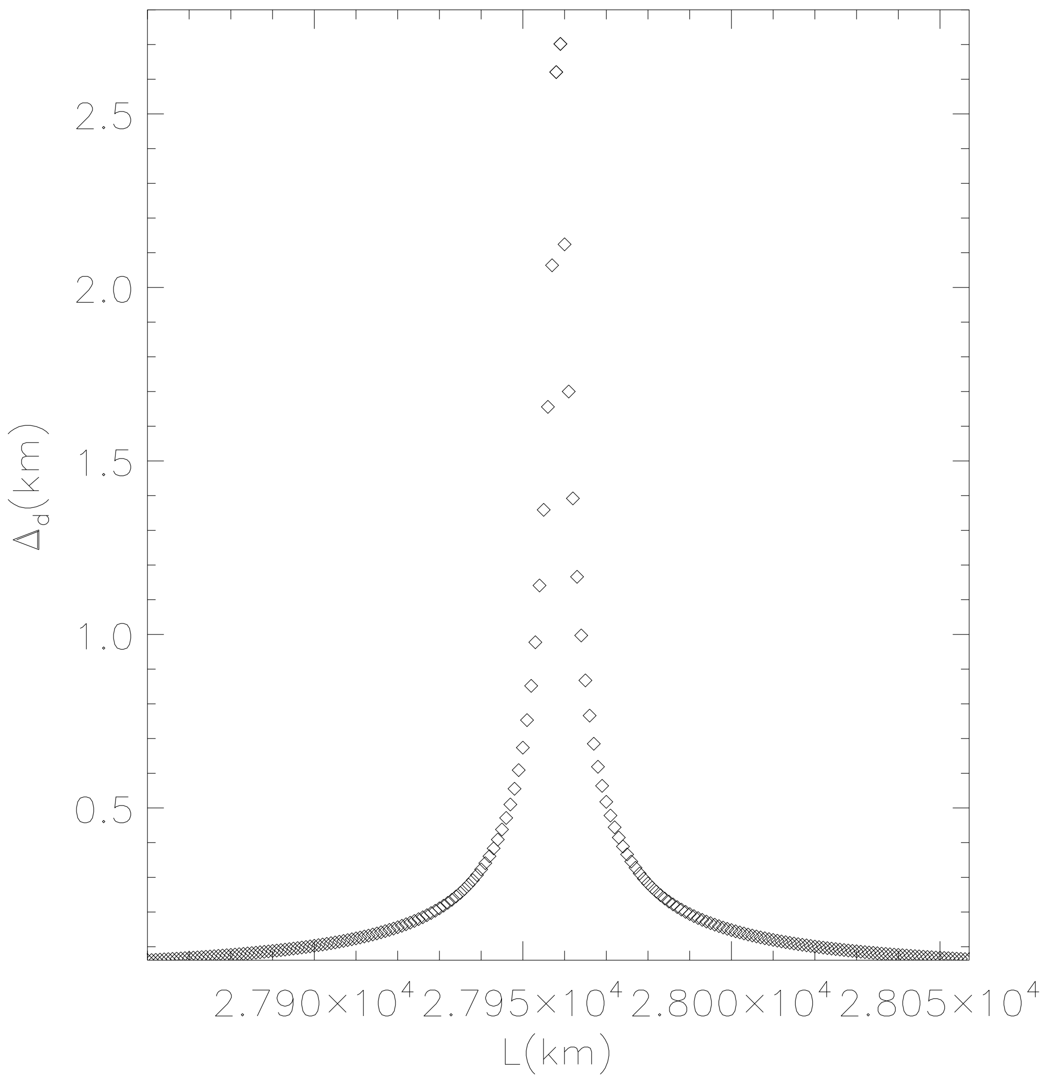
($c^\star$): Finally, by using the analytical solution described in previous talk, new inertial coordinates $x^\alpha + \Delta(x^\alpha)$ may obtained, from the perturbed emission coordinates $\tau^A + \Delta(\tau^A)$. Coordinates $x^\alpha + \Delta(x^\alpha)$ are to be compared with the coordinates x^α obtained in (c).

Quantity $\Delta_d = [\Delta^2(x^1) + \Delta^2(x^2) + \Delta^2(x^3)]^{1/2}$ is a good estimator of the positioning errors produced by uncertainties ξ_A^α in the satellite motions.

For a certain direction, we have taken an interval of $200\ Km$ centered in a zero of J and, then, quantity Δ_d has been calculated in 200 uniformly distributed points in the chosen interval around $J = 0$.

For each of the 200 points, the same random deviations ξ_A^α have been used to perturb the satellite world lines (with an amplitude of $1\ m$ for every satellite). Results are shown in next slide:

POSITIONING ERRORS: PRELIMINARY RESULTS I



Estimator Δ_d (in Km) against the distance L to E (in Km) along the chosen direction (in the $200\ Km$ interval). Close to the point where $J = 0$, quantity Δ_d takes on very large values

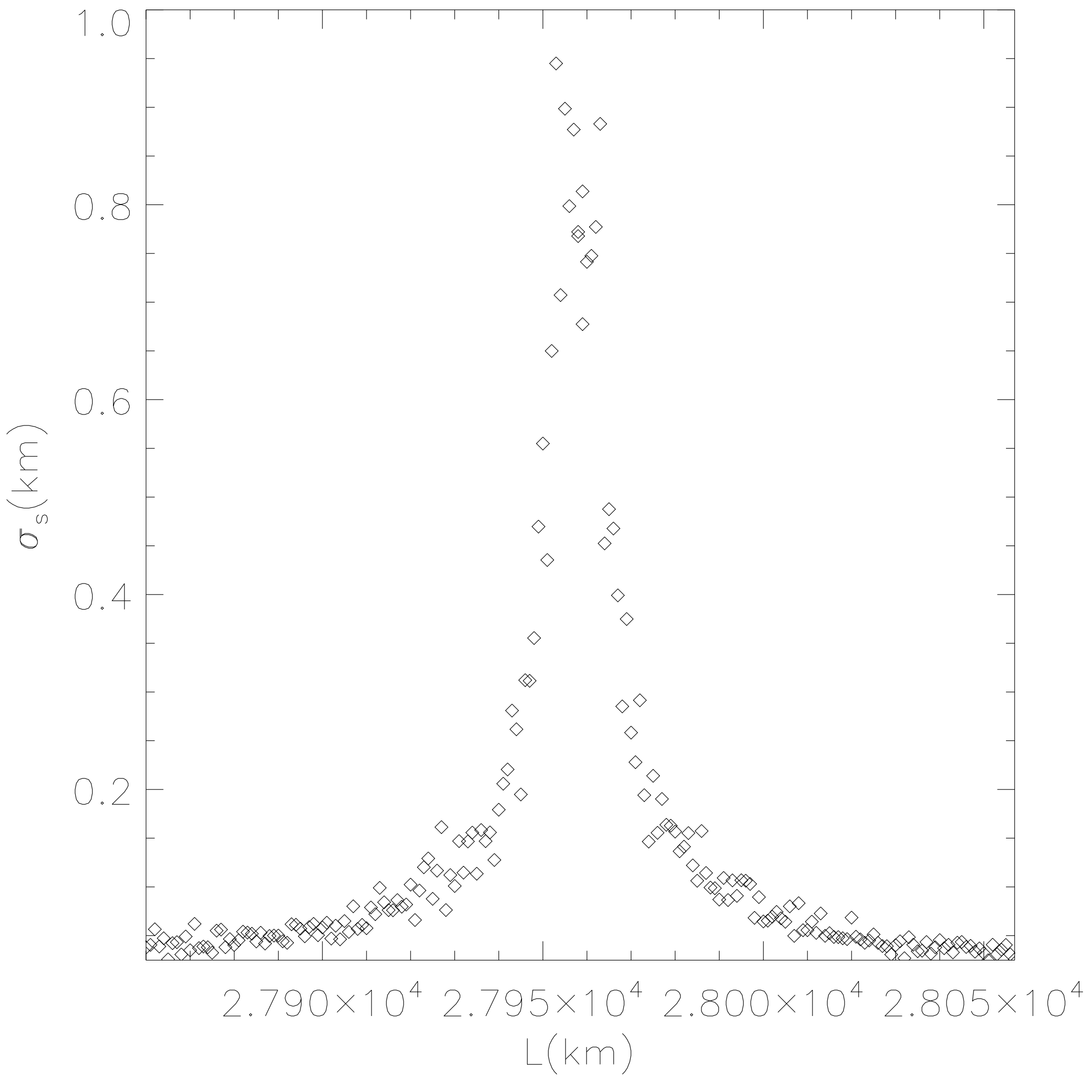
POSITIONING ERRORS AND SATELLITE UNCERTAINTIES II

In order to see that positioning errors are large, around the point where $J = 0$, for any choice of ξ^{α}_{A}, a preliminary statistical study has been performed.

For each point of the chosen interval, ten random deviations ξ^{α}_{A} have been generated around each satellite (with a maximum amplitude of $1\ m$ in all cases). These deviations have been combined among them to get 10^4 cases. Quantity Δ_d has been computed in each of these cases.

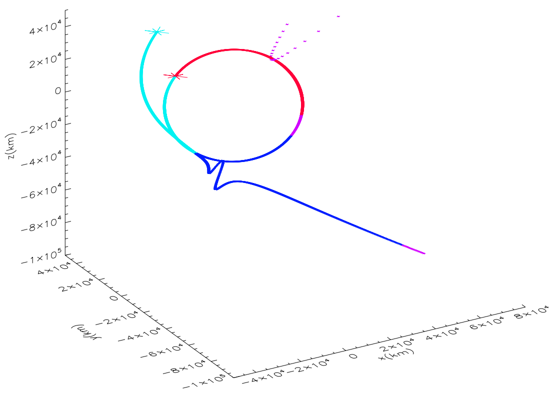
The rms (root mean square) value σ_s of the 10^4 quantities Δ_d has been obtained in each point. Results are presented in next slide:

POSITIONING ERRORS: PRELIMINARY RESULTS II

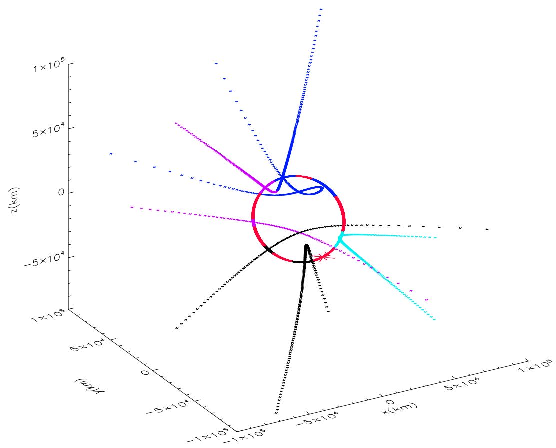


Standard deviation σ_s (in Km) against the distance L to E (in Km) in the chosen $200\ Km$ interval. Close to the point where $J = 0$, typical positioning errors are very big

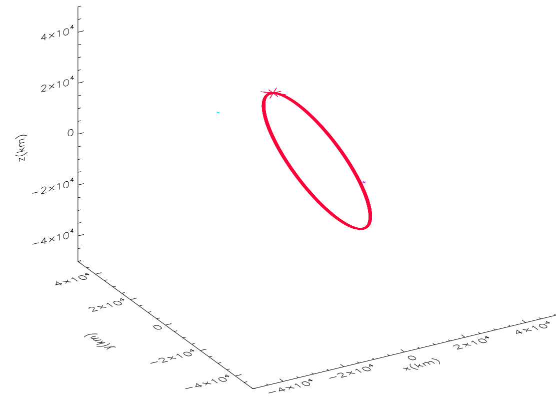
GALILEO-GPS AND GPS-GALILEO



(n)



(ñ)



(o)

(n) and ñ: Positioning a GALILEO satellite (receiver) with four GPS emitters.

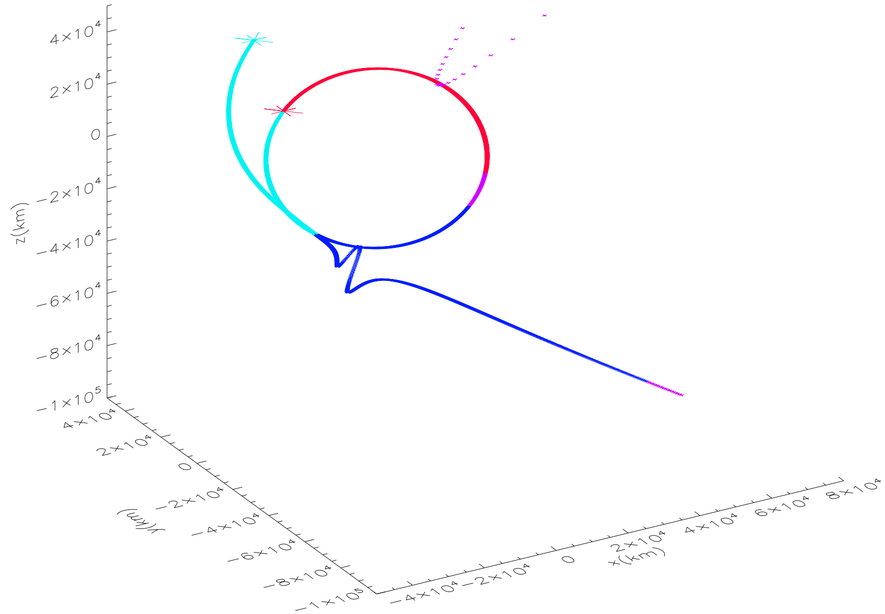
(o): Positioning a GPS satellite (receiver) with four GALILEO emitters.

7200 equally spaced points are considered on the receiver world line. In each point, the emission coordinates are calculated (Newton-Rhapson) and, from them, the sign of χ^2, which tell us if the point has an associated false point (bifurcation) or it is single valued.

Single valued points are red and they are located on the receiver circumference.

Four sets of 1800 points have been selected and –in the case of bifurcation– these sets have been ordered in the sense of growing time (dextrogyre) by using the following sequence of colors: black, fuchsia, dark blue and light blue.

GALILEO-GPS

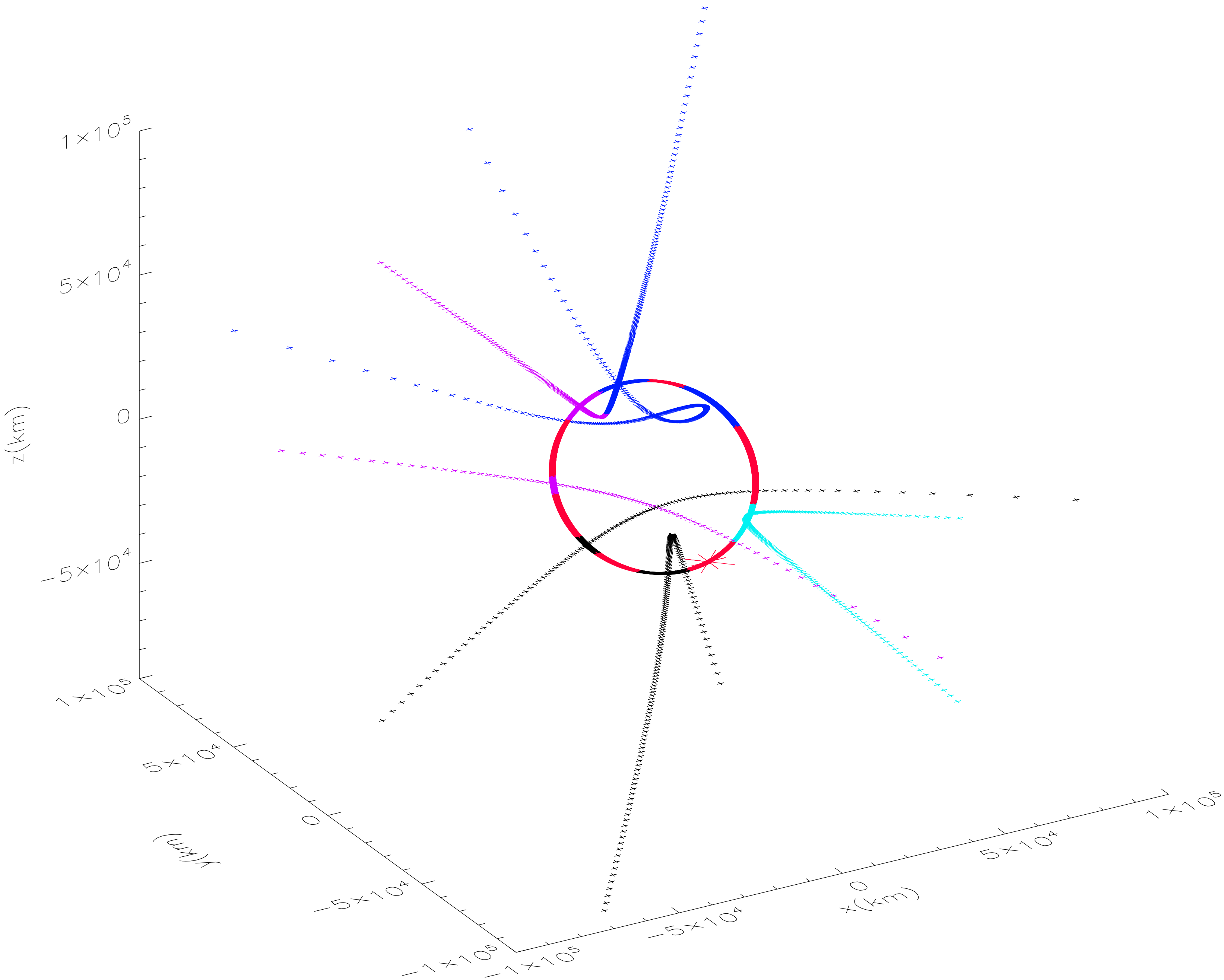


Initial points may be: a single valued red point (represented by a star) or a bifurcation represented by two black stars.

Since GPS and GALILEO satellites have not the same period, the final point has not always the same χ^2 sign as the initial one. In the case of bifurcation one of the points is on the circumference and the other point is an external light blue star.

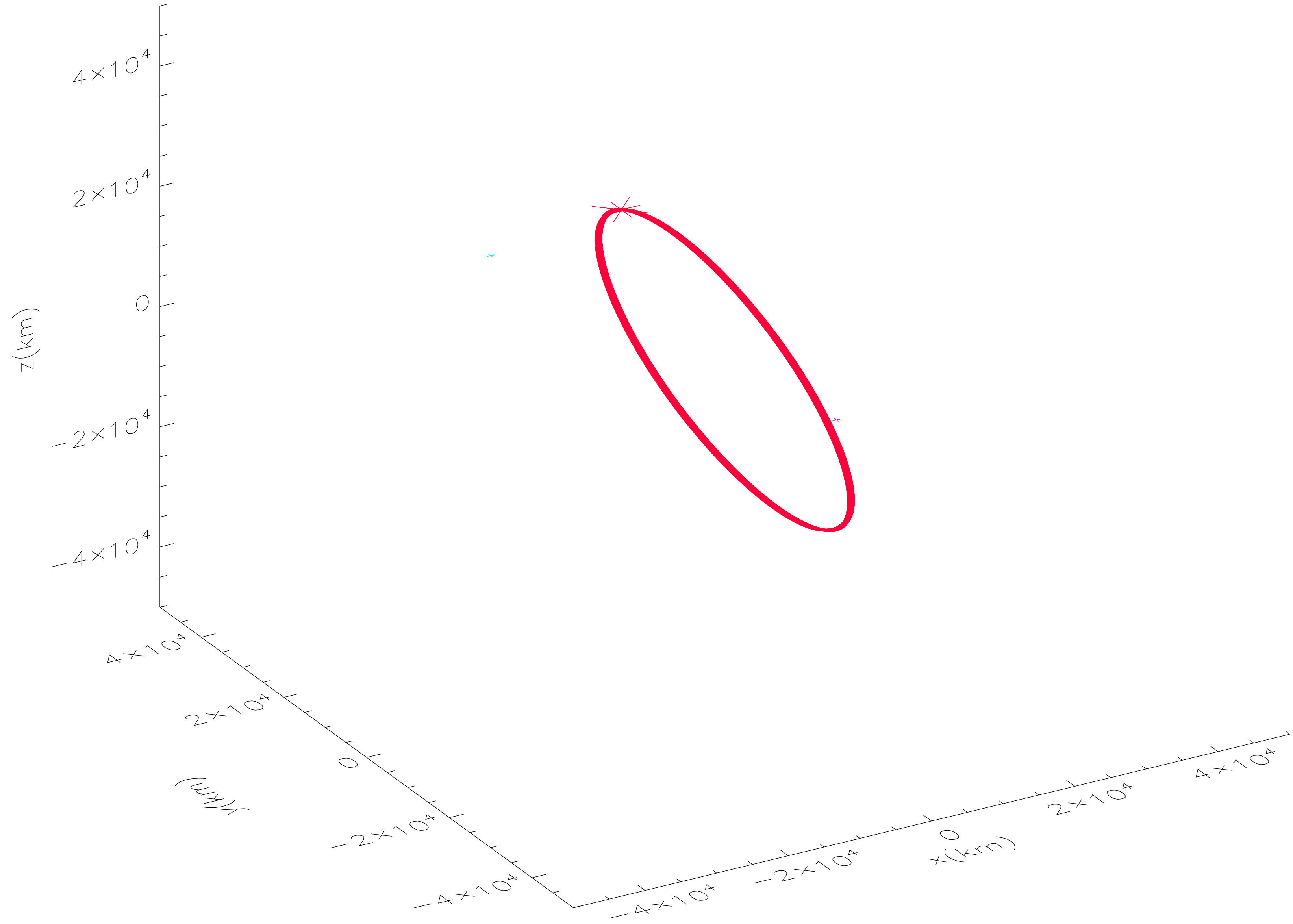
In the transition from red (single solution) to any other color (double solution), one of the positions is on the circumference and the other one tends to infinity. The same occurs from any color (bifurcation) to red (single). Any other color change is continuous (we have decided to change the color to follow the satellite motion):

GALILEO-GPS



1×10^5
5×10^4
0
−5×10^4
z(km)
1×10^5
5×10^4
0
−5×10^4
−1×10^5
y(km)
−1×10^5
−5×10^4
0
x(km)
5×10^4
1×10^5

GPS-GALILEO



GENERAL DISCUSSION I

- In our approach, satellites move in Schwarzschild space-time, so the effect of the Earth gravitational field on the clocks is taken into acount, e.g., GPS clocks run more rapid than clocks at rest on Earth by about 38.4 microseconds per day. It has been verified and taken into account.
- Photons have been moved in Minkowski space-time. It is due to the fact that the Earth gravitational field produces a very small effect on photons, while they travel from the satellites to the receiver. The distance travelled is not large and the gravitational field is weak.
- We may obtain the emission coordinates from the inertial ones by using accurate numerical codes based on the Newton-Raphson method.
- We may obtain the inertial coordinates from the emission ones by using the analytic transformation law of Coll, Ferrando & Morales-Lladosa.
- From the emission coordinates and the satellite world lines, one easily known if there is only a possible receiver position or two (bifurcation). In the second case, Morales-Ladosa has proposed a method (angle measurements) two select the true position. Other methods are possible (see, paper I).

GENERAL DISCUSSION II

- Small uncertainties in the satellite world lines produce large positioning errors if $J \simeq 0$.
- If possible, the four emitters (satellites of the chosen GNSS) must be selected to avoid both bifurcation and situations with $J \simeq 0$.
- Positioning on Earth surface is always single ($\chi^2 \leq 0$) and the Jacobian does not vanish in this case. Hence, our study applies to the case of objects located away from Earth surface.
- We are moving photons in the case of Schwarzschild, Kerr, and PPN metrics; however, only small corrections arise with respect to the approach assumed here. Previous work on this subject has been performed by various authors (see references in paper I).