“GPS” observables in General Relativity

Carlo Rovelli
Ljubjana, September 2012

An idea at the roots of the *Relativistic Positioning Systems* :

Navigation (GNSS) technology suggests an elegant solution to a long standing problem in theoretical general relativity

C Rovelli: *GPS observables in general relativity*, gr-qc/0110003.

Similar idea (independently): *Bartolomé Coll* (2001: Proc. 23rd ERE Spanish Relativity Meeting, Reference Frames and Gravitomagnetism).
Developed by: *Pozo, Tarantola, Lachièze-Rey, Morales, Ferrando, Bini, Lusanna, Mashoon*, and others.

The name of the game: GPS coordinates, GNSS coordinates, ABC coodrinates, Emission coordinates, Radar coordinates ...

GNSS systems $\longleftarrow$ General relativity

What do they teach one another?

- Relativistic corrections

- Overcome viewing relativistic effects as corrections over Newtonian spacetime: conceiviving GNSS in a fully *general relativistic* conceptual framework? (cfr: no separation geodesic/time reference system...)

- also ... thinking ahead to extraterrestrial navigation.

GNSS systems $\longmapsto$ General relativity

What do they teach one another?

- What are the *observables* quantities?
- Can we write a *complete* set of observables?
- Is there a (realistic) way to *identify spacetime points*?

GNSS systems $\longmapsto$ General relativity

The problem:

- In GR observable (gauge invariant) quantities are coordinate independent.

(Exemple: the Mars-Earth *distance* at sunrise this morning.)

- Can we write down a *complete* set of observable quantities?

E.M.:

- Field: $A_\mu(x)$
- Gauge invariant quantities: $F_{\mu\nu}(x)$

R.G.:

- Field: $g_{\mu\nu}(x)$
- Gauge invariant quantities:  ?

The phase space of relativity is not the space of the Einstein metrics on a manifold, but a much smaller space:

$$Geom_4 = \frac{Metrics_4}{Diff_4}$$ (pure gravity)

$$\Gamma = \frac{Metrics_4 \times Matter}{Diff_4}$$ (with matter)

Gauge invariant observables are functions on Γ.

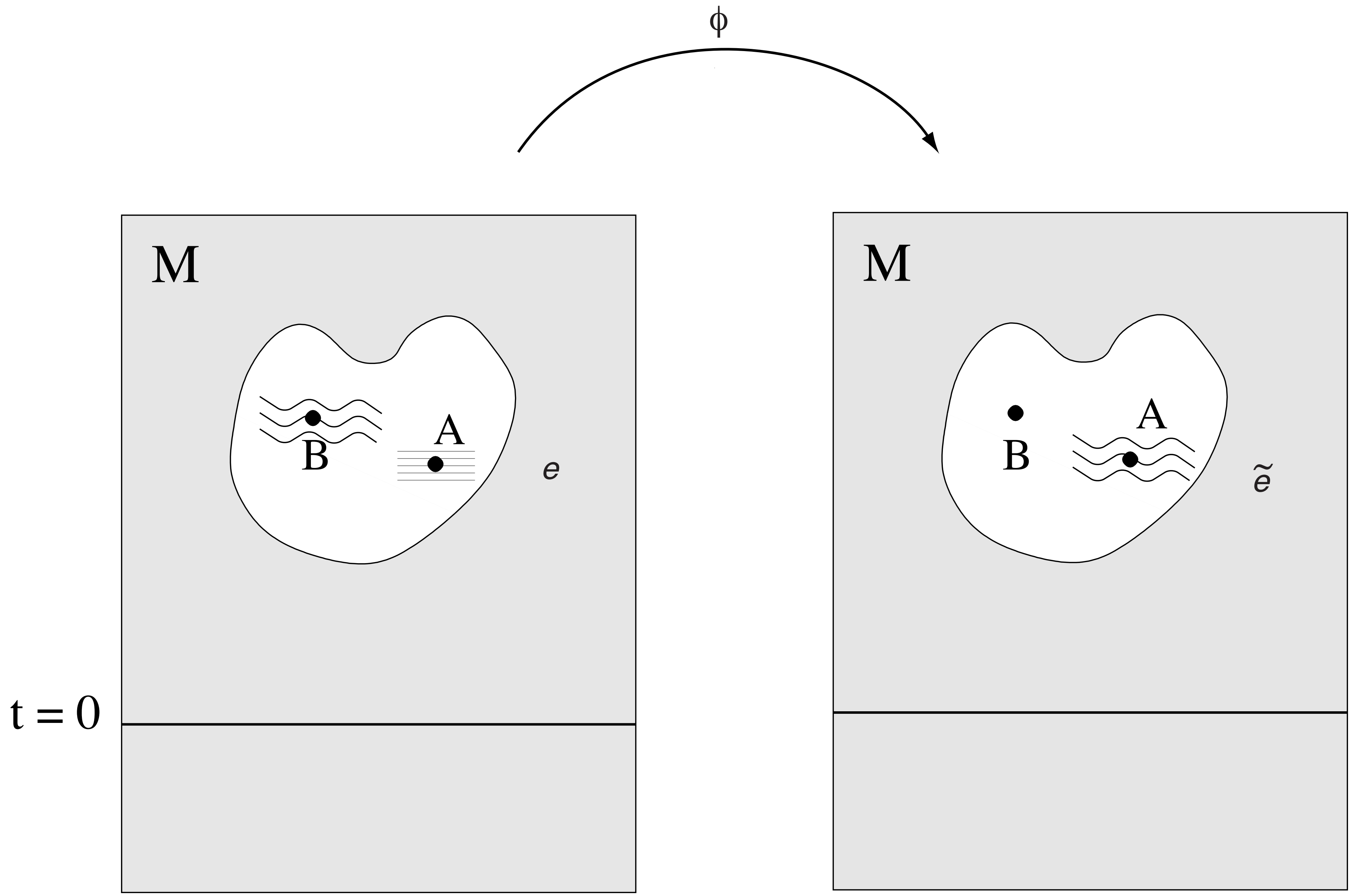
What are these? How to coordinatize Γ ?

General relativity "takes away from space and time
the last remnant of physical objectivity"

A Einstein, *The foundations of general relativity*, 1916

"On the basis of general relativity, space, as opposed to "what fills space",
has no separate existence.
If we imagine the gravitational field to be removed,
there does not remain a space [...],
but absolutely nothing.

A Einstein, *Relativity, the special and general theory*, 1917

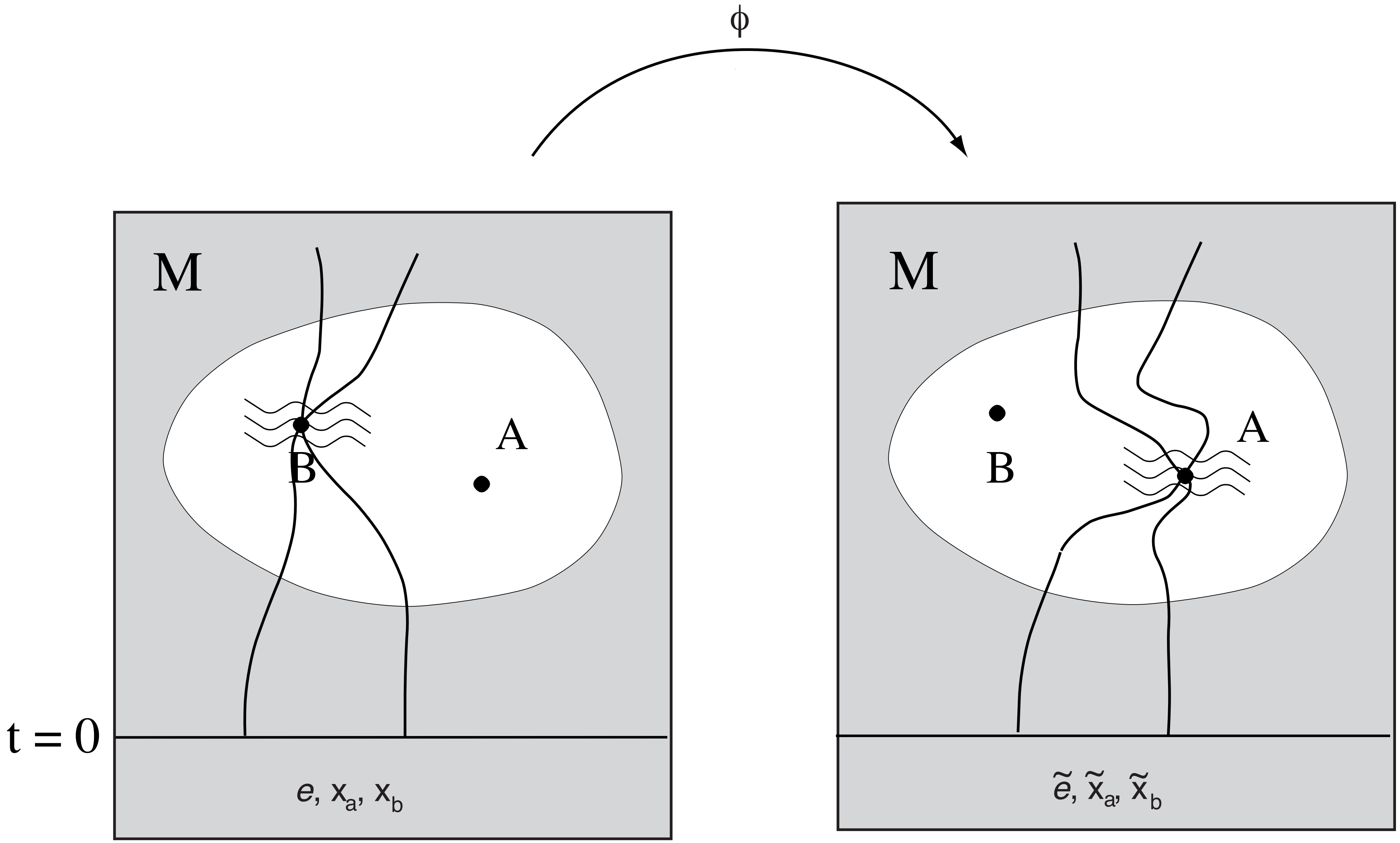


"What happens at point B"
is a *meaningless* question

"All our space-time verifications invariably amount
to a determination of space-time coincidences.

If, for example, events consisted merely in the motion
of material points, then ultimately nothing would be observable
but the meetings of two or more of these points"

A Einstein, *The foundations of general relativity*, 1916



“What happens at point where two particles meet”
is a *meaningfull* question

Previous solutions:

- Use four curvature invariants (R, $R_{\mu\nu} R^{\mu\nu}$, , ...) as coordinates. (*Bergmann. Komar...*).

- Fill spacetime with matter (cosmology): dust, four fictitious scalar field ..., use matter to localize points. Use one ideal clock on each point to define the time coordinate. (*DeWitt, CR, Brown-Kuchar, Marolf, Thiemann, Lewandoski, ...*)

- These solutions are unrealistic, impractical outside limiting situations, or technically nearly impossible to deal with.

The solution suggested by the GNSS:

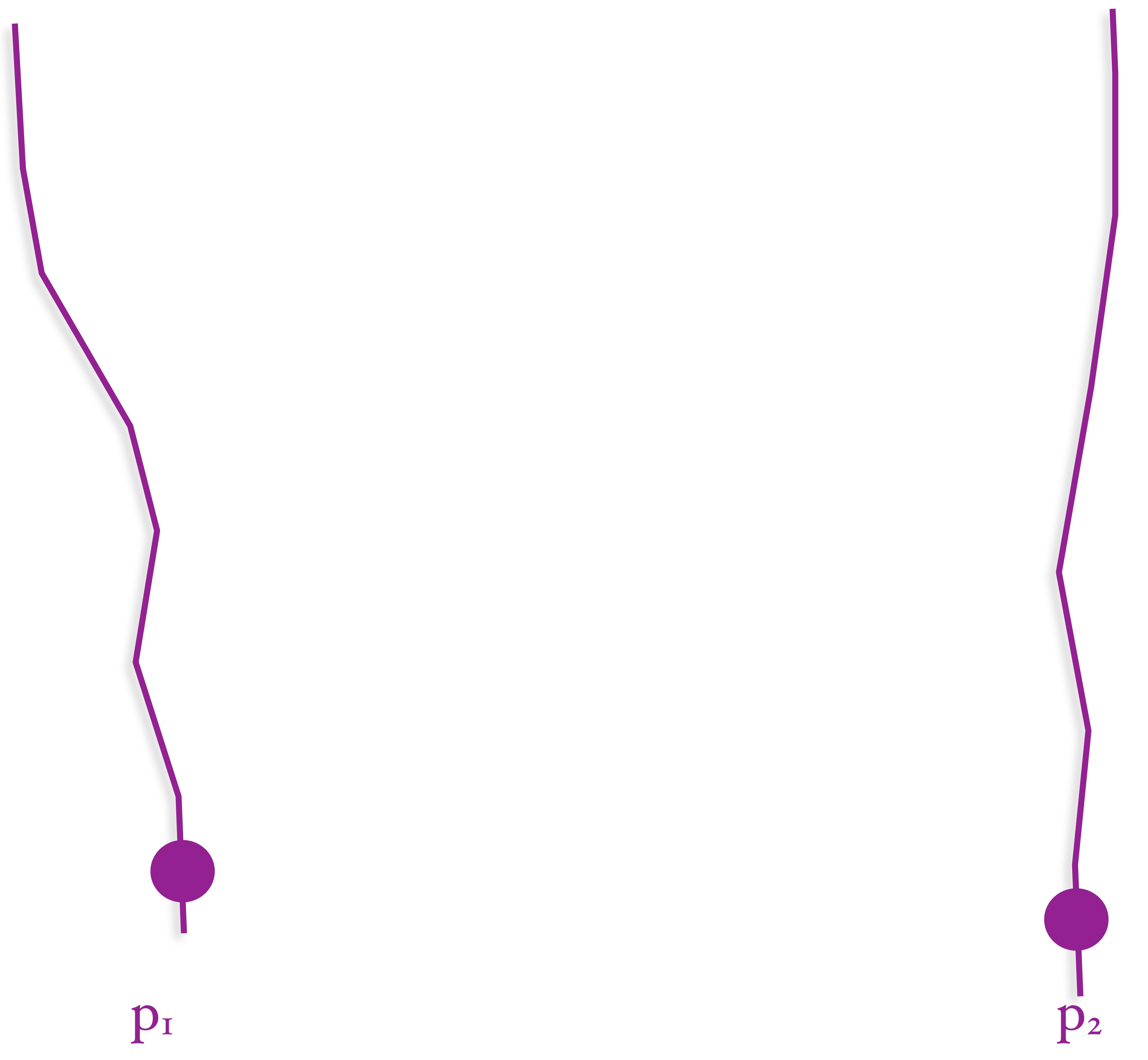
- Consider General relativity coupled with four "particles" p_1, p_2, p_3, p_4.

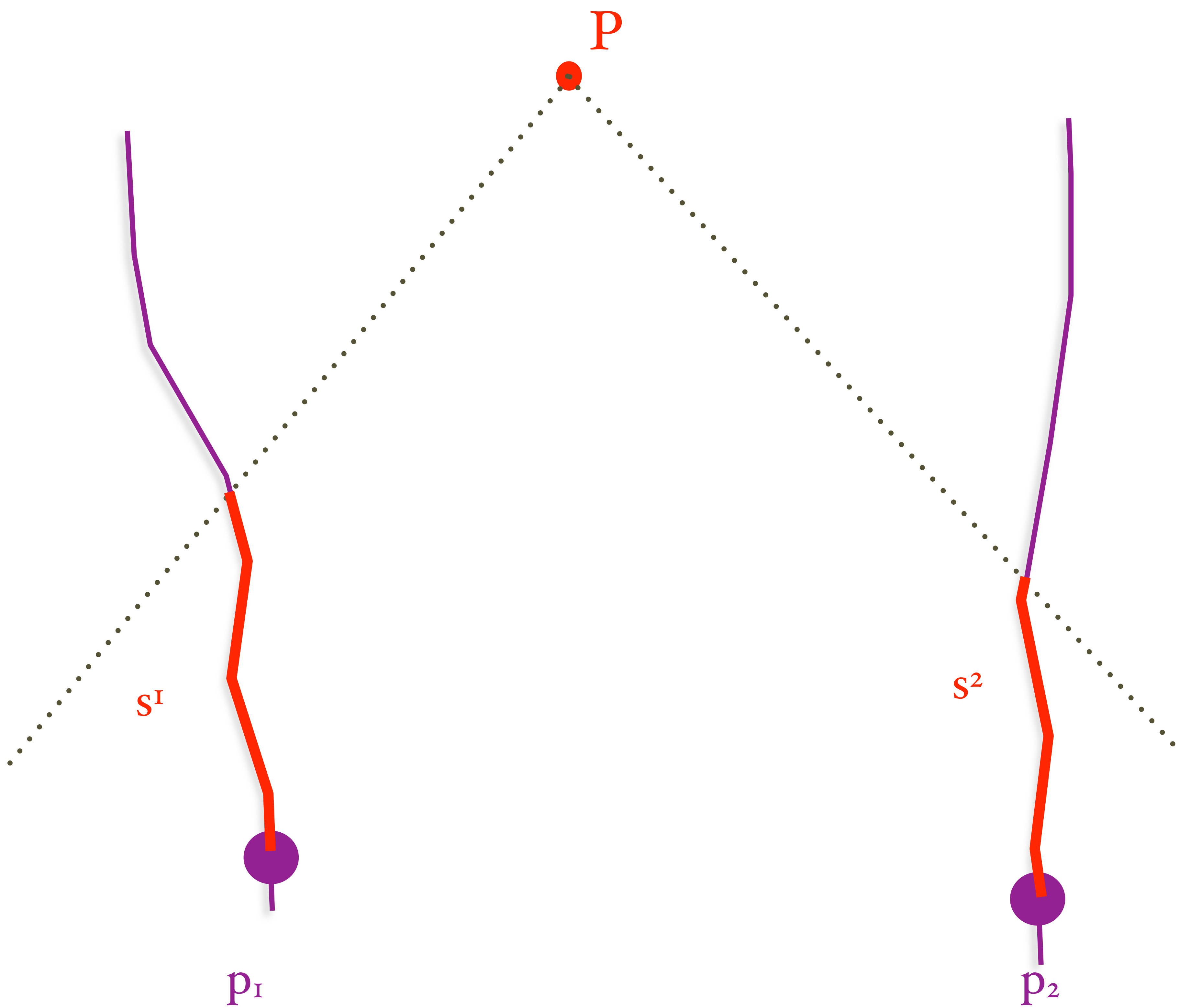
  Assume a point ("origin") is marked along the four worldlines l_1, l_2, l_3, l_4 of of these four particles.

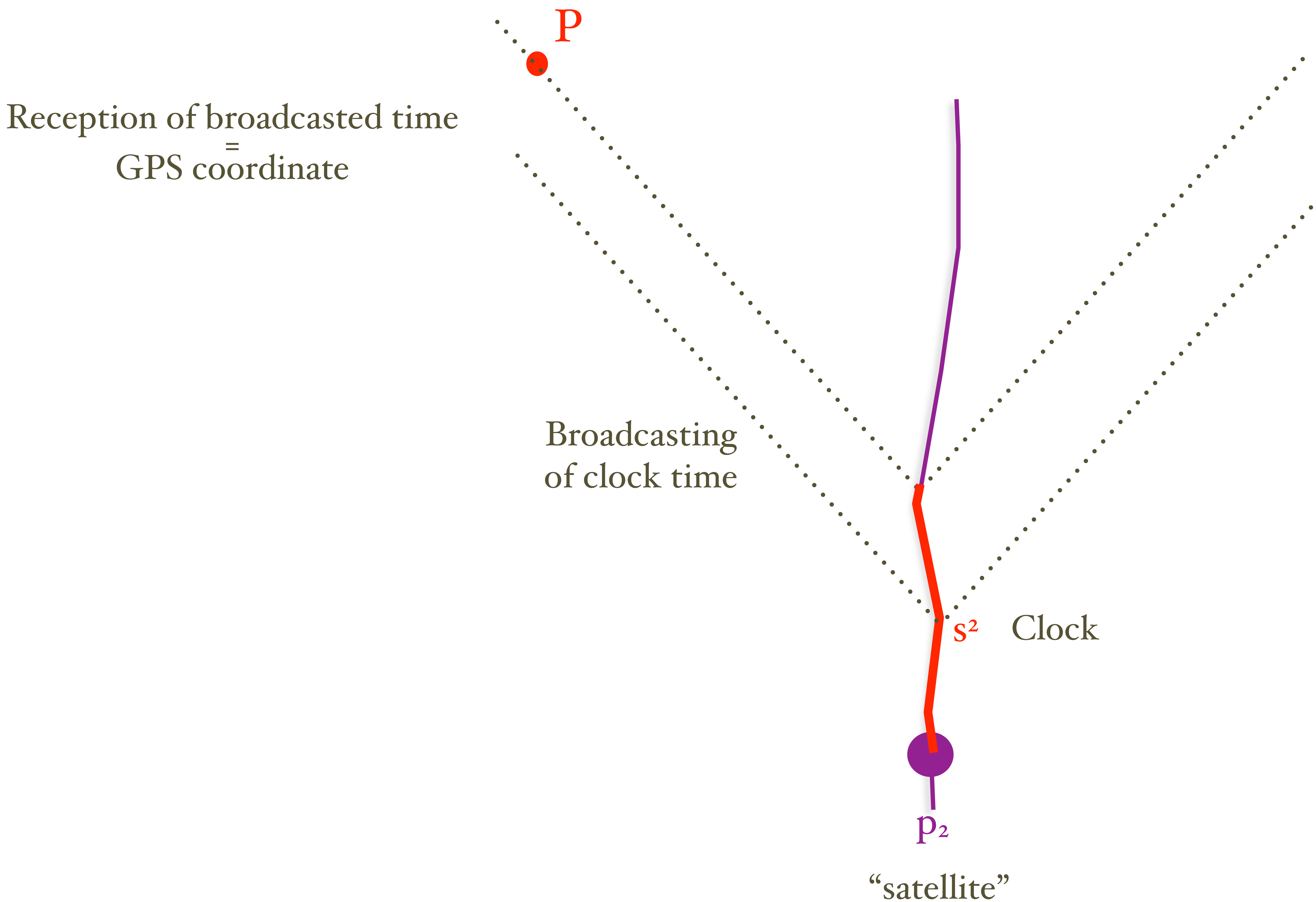
- Then there is immediately a natural reference system in spacetime defined as follows:

  - Given a point P in spacetime, consider its past light cone. Generically, it intersects the worldline l_i once. Let (s^1, s^2, s^3, s^4) be the physical lengths of the portion of the worldlines between the origin and the intersection with the past light cone of P .

  - Then (s^1, s^2, s^3, s^4) are natural coordinates for P .


p1
p2


P
s¹
s²
p₁
p₂


P
Reception of broadcasted time
=
GPS coordinate
Broadcasting
of clock time
s²
Clock
p₂
“satellite”

Thus:

- Four “objects” coupled to general relativity (subjected to arbitrary forces) are sufficient to define a natural coordinatization of all spacetime points.
- This coordinatization is easy to implement realistically: in fact, it is physically *realized* by the GNSS systems.

Observables:

- The components of the metric tensor expressed in GPS coordinates $s = (s^1, s^2, s^3, s^4)$ are physical observables :
- How to measure a component of $g_{\mu\nu}(s)$:

$$g_{11}(s) = (L / \Delta s_1)^2$$

- This provides a simple effective way to map the metric tensor in a spacetime region.

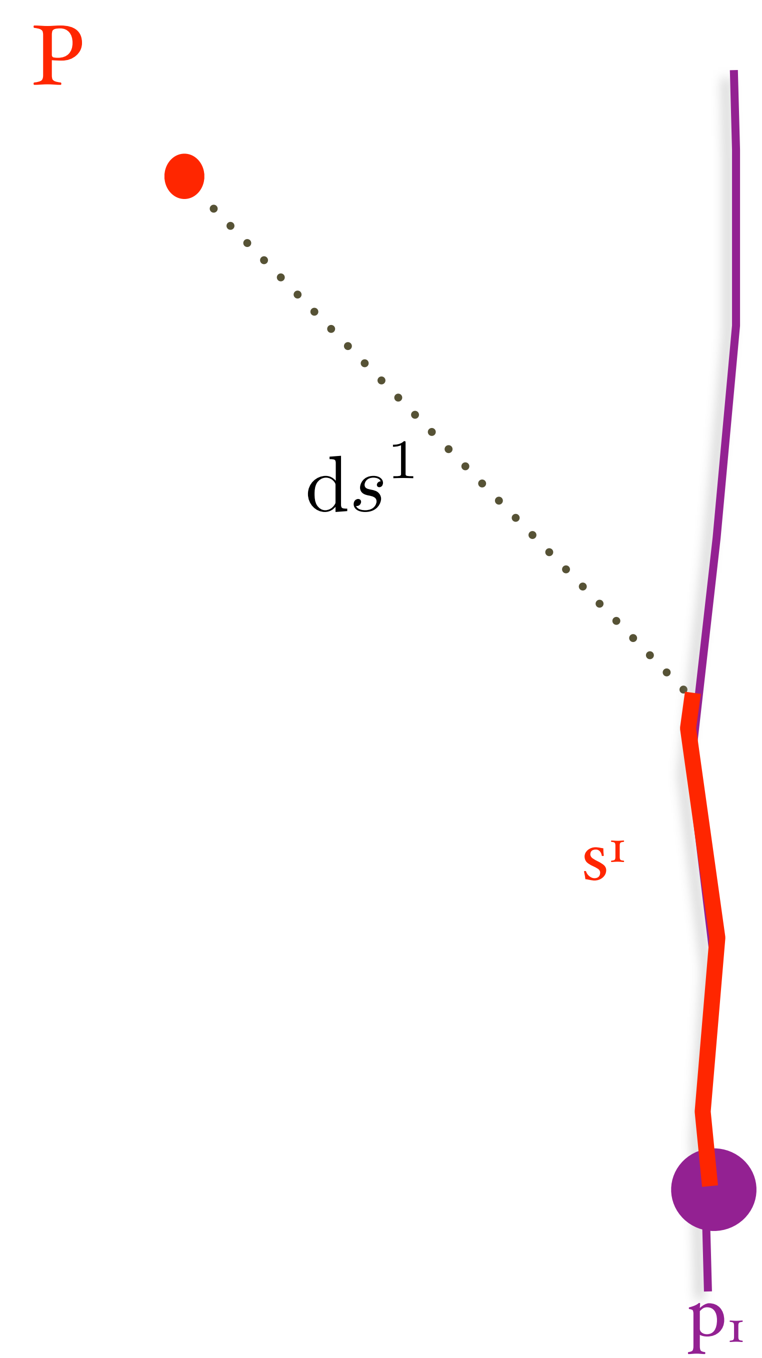
Some crucial properties of the GPS coordinate system:

- $g^{\alpha\alpha} = 0$

  Because $\mathrm{d}s^\alpha$ (equal coordinate surface) is a null surface.

$$\mathrm{d}s^\alpha = \delta^\alpha_\beta \; \mathrm{d}s^\beta$$

$$|\mathrm{d}s^\alpha|^2 = g^{\beta\gamma}\delta^\alpha_\beta\delta^\alpha_\gamma = g^{\alpha\alpha}$$



Some crucial properties of the GPS coordinate system:

- Evolution equations for the components of the metric tensor are completely *local* !

$$\begin{aligned} ds^2 &= g_{\mu\nu} dx^\mu dx^\nu \\ &= N^2 dt^2 - \gamma_{ab}(dx^a - N^a dt)(dx^b - N^b dt). \end{aligned}$$

$$g^{\mu\nu} v_\mu v_\nu = -\gamma^{ab} v_a v_b + (n^\mu v_\mu)^2 \qquad n^\mu = (1/N, N^a/N)$$

$$W^{\alpha\mu} = \frac{1}{\sqrt{1-v^2}} (1,\ v\, v^{\alpha a}). \qquad W_a^\alpha W_b^\alpha \gamma^{ab} = (W_\mu^\alpha n^\mu)^2$$

$$N = \frac{1}{W_\alpha^0 q^\alpha}, \qquad N^a = \frac{W_\alpha^a q^\alpha}{W_\alpha^0 q^\alpha}.$$

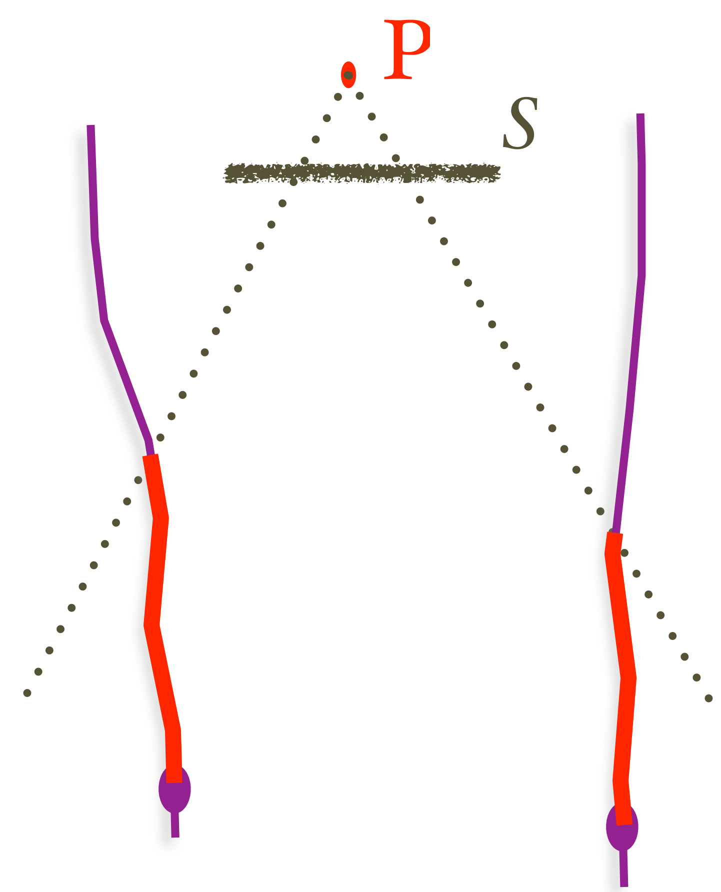
Lapse and Shift can be solved *locally as functions of the metric.*

Some crucial properties of the GPS coordinate system:

- Evolution equations for the components of the metric tensor are completely *local* !

How is it possible?

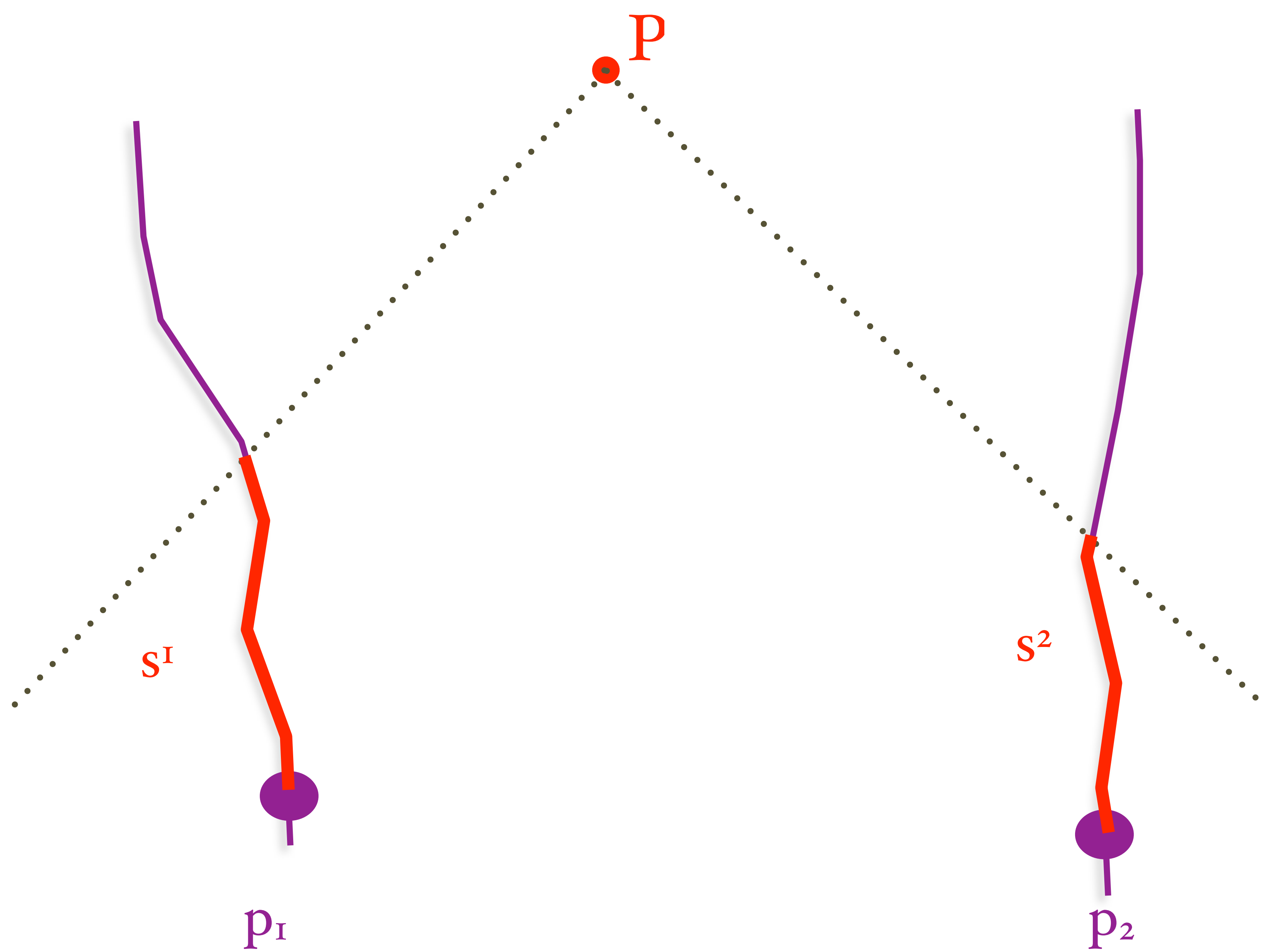
Because the local coordinates are determined by what happens far away, but right in the *past* of the point. If P is the past domain of dependence of S then what happens around P is fully determined by the fields on S.



Some crucial properties of the GPS coordinate system:

- GPS coordinates realize concretely Einstein's idea that all physical observations, and the localization of spacetime points, can only be based on "point coincidences" of particle trajectories: here we have point-coincidences of photons' trajectories.

"... ultimately nothing would be observable but the meetings of two or more of these points.."
AE



GNSS systems $\longleftarrow\!\!\!\dashv$ General relativity

Can all this be useful for navigation ?

- ☑ A natural reference system is defined. No Newtonian concept. No clock synchronization, no distances. No space/time separation. No assumption whatsoever about the form of the metric. It works in principle in any arbitrary gravitational field.

- ☑ It only requires four objects to be specified. It is then immediately implemented. → Four GNSS stations?

- ☑ It is local.

Potential utility for GNSS

☑ A completely autonomous system: the satellites can define and use a reference system without need of stations on Earth.

☑ Solar system and deep-space navigation.

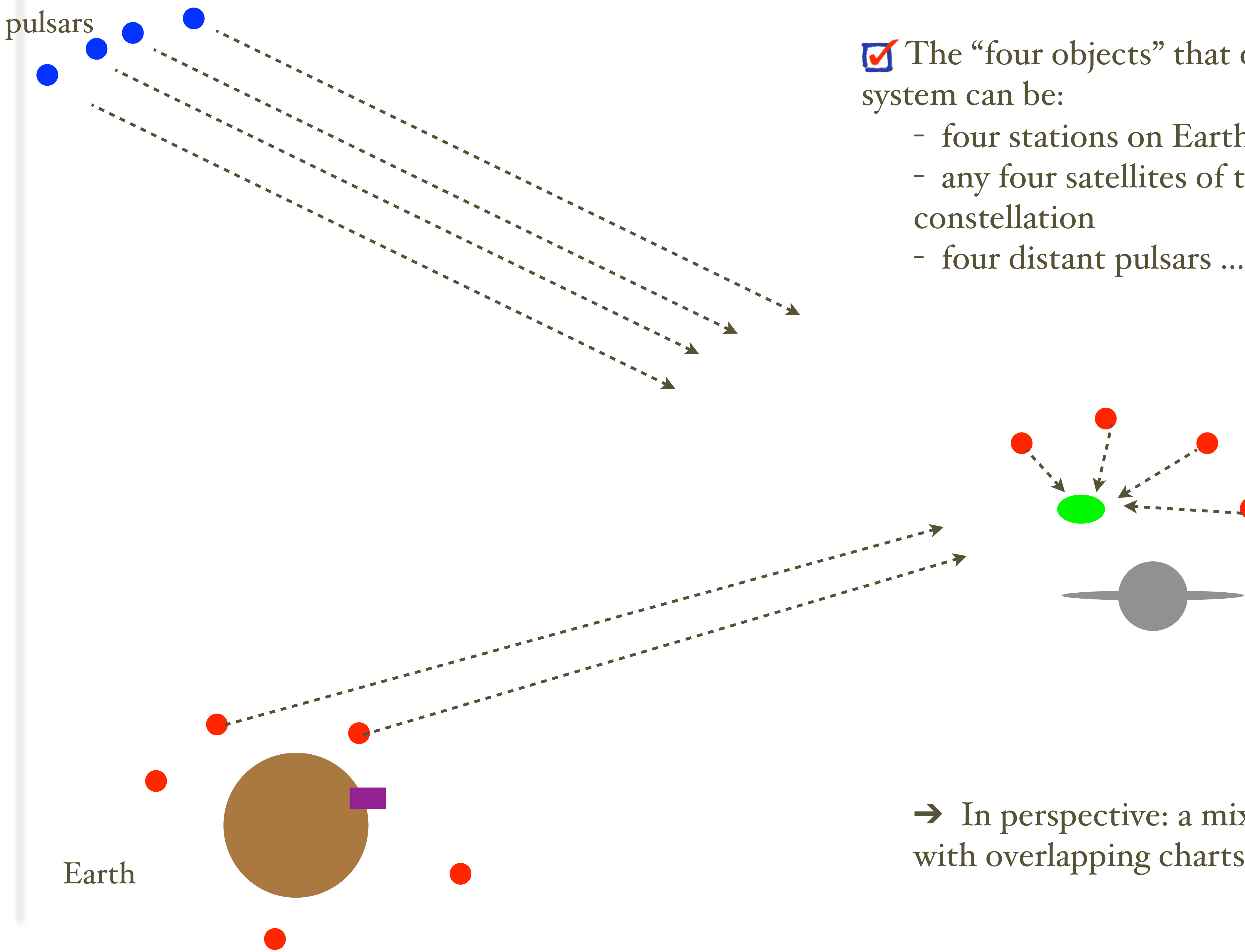
☑ Potentially useful in arbitrary time dependent

☑ A navigation system based on our true understanding of spacetime.

☑ The “four objects” that define the system can in principle be:

- four stations on Earth.
- any four satellites of the constellation
- four distant pulsars …

➔ In perspective: a mixed system, with overlapping charts ?



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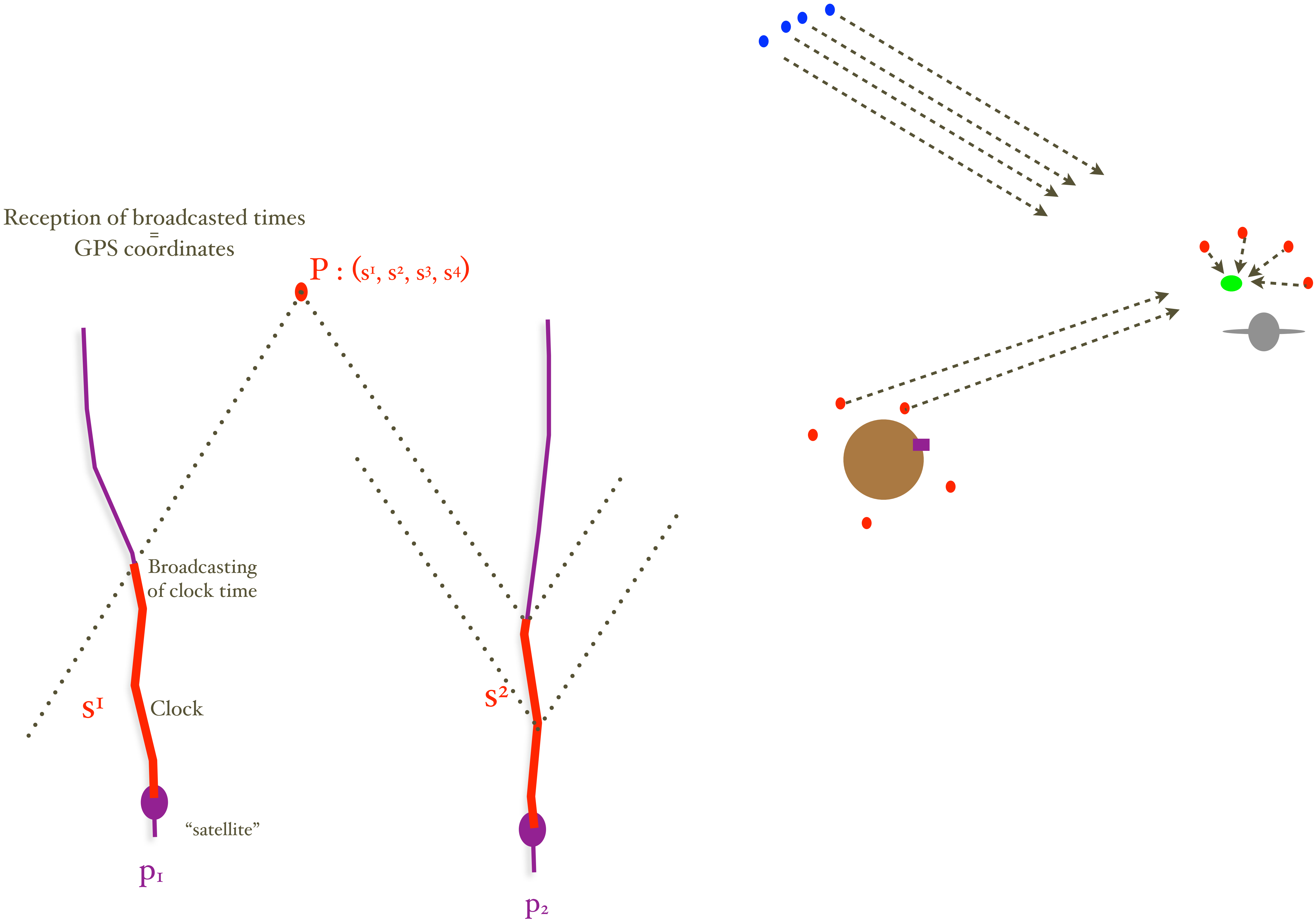
Main technical complication (to be studied)

- The transformations equations between the reference systems defined by two different quadruplet of objects are complicated, if the metric is complicated.

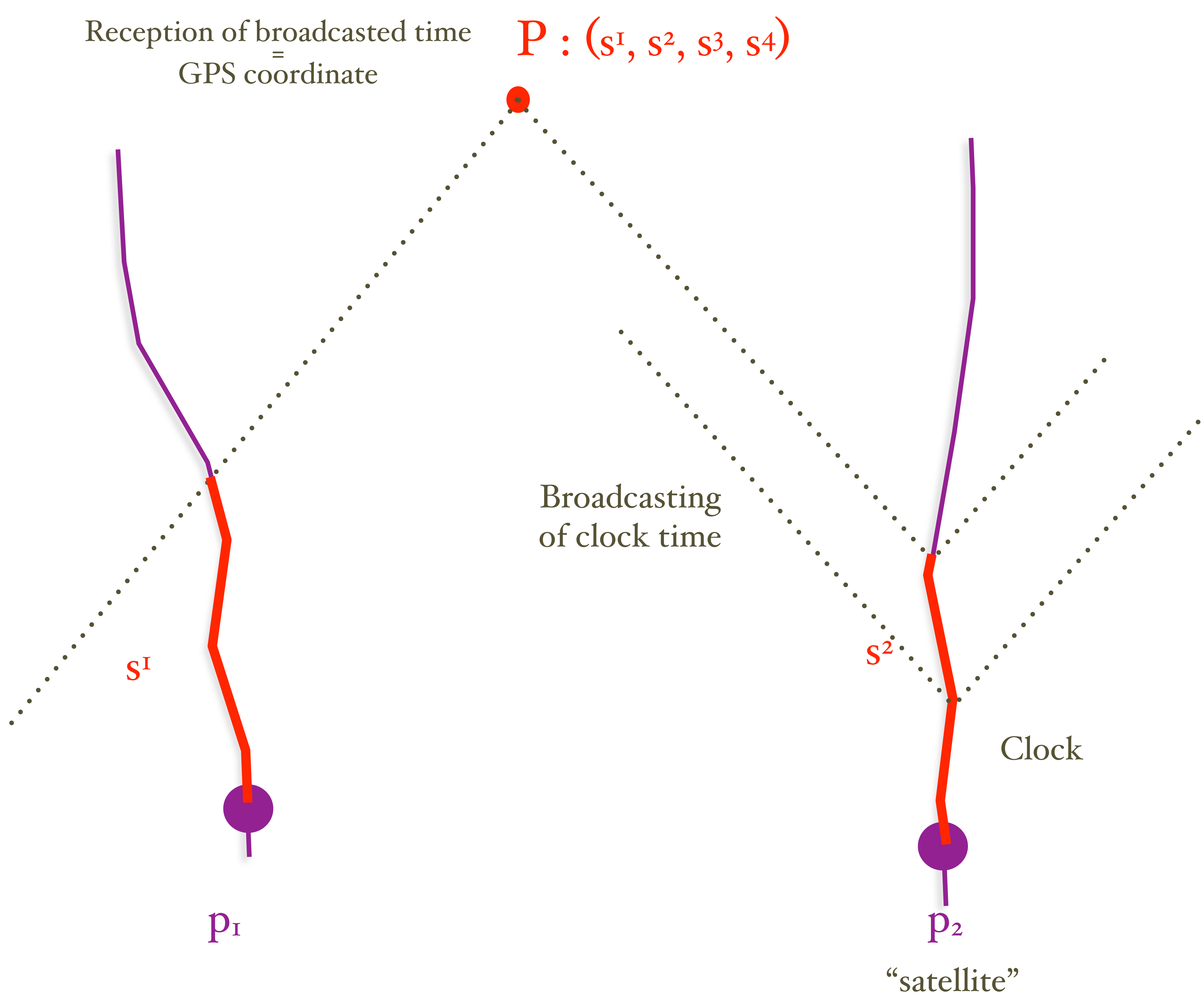
(In Minkowski space, with four (α=1,2,3,4) reference particles moving along straight lines at 4-velocity $V^\mu{}_\alpha$, the GNSS coordinates are related to Minkowski coordinates x^μ by

$$s^\alpha = \vec{X} \cdot \vec{W}^\alpha - \sqrt{(\vec{X} \cdot \vec{W}^\alpha)^2 - |\vec{X}|^2}$$

)



Reception of broadcasted times
=
GPS coordinates
P : (s¹, s², s³, s⁴)
Broadcasting
of clock time
s¹
Clock
"satellite"
p₁
s²
p₂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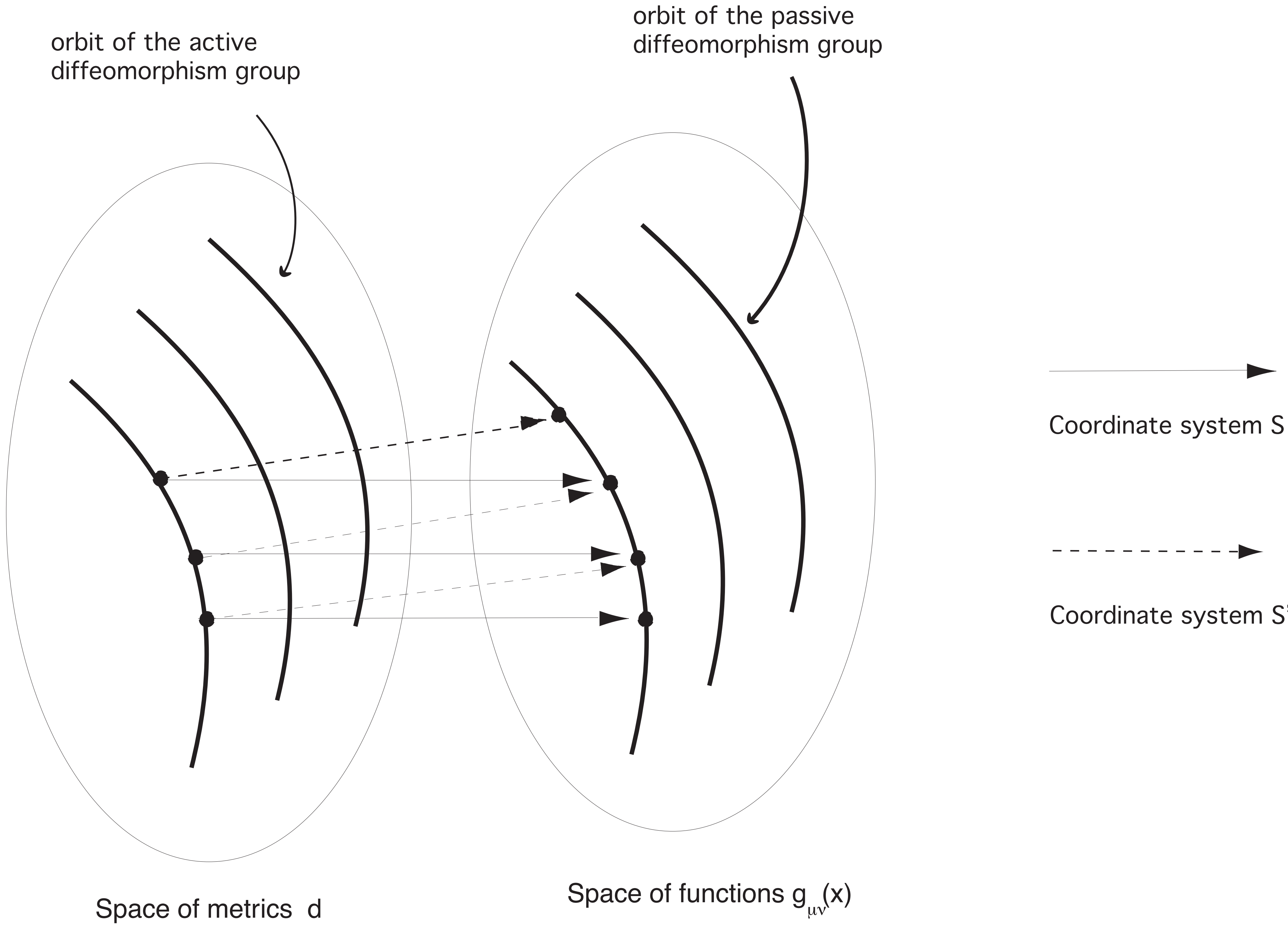
orbit of the active
diffeomorphism group
orbit of the passive
diffeomorphism group
Coordinate system S
Coordinate system S’
Space of metrics d
Space of functions $g_{\mu\nu}(x)$