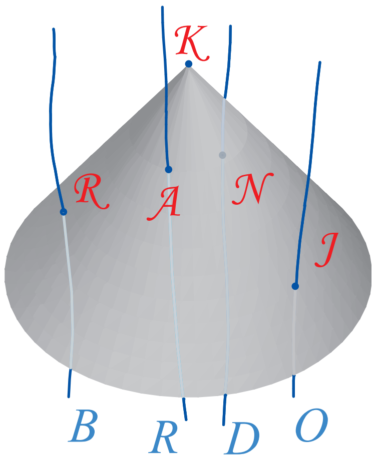
Workshop Relativistic Positioning Systems and their Scientific Applications

Brdo near Kranj, Slovenia, 19-21 September 2012

From emission to inertial coordinates: Splitting of the solution relatively to an inertial observer



J. A. Morales-Lladosa

Departament d'Astronomia i Astrofísica

Universitat de València

In collaboration with B.Coll and J. J. Ferrando

Outline

We consider Relativistic Positioning Systems (RPS) in Minkowski space-time, focussing the attention on:

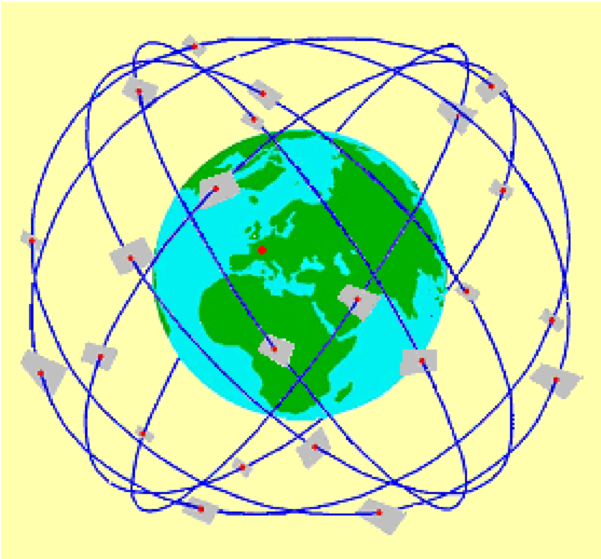
(i) the location problem: to determine the inertial coordinates of a user from a standard set of data,

(ii) the bifurcation problem: to choose the true solution of the location problem when it admits two formal solutions,

(iii) the space and time splitting of the quantities providing the user location.

Location problem in Global Navigation Satellite Systems (GNSS)

GNSS



Satellite constellation

Emission equations:

$$c(t - t^A) = |\vec{x} - \vec{\gamma}_A(t^A)|$$

$$A = 1, 2, 3, 4, \ldots$$

Data: $\{t^A, \vec{\gamma}_A(t^A)\}$
(in a specific terrestrial frame)

Unknowns: $(t, \vec{x})$ user location

- Iterative procedures.
- Numerical algorithms.
- Exact (and covariant) expression.

Confronting numerical and analytical approaches

$$c(t - t^A) = |\vec{x} - \vec{\gamma}_A(t^A)|, \quad A = 1, 2, 3, 4$$

- Iterative procedures:

$$\vec{x}_0 \underset{A=4}{\longrightarrow} t = t_0 \underset{A=1,2,3}{\longrightarrow} \vec{x}_1 \underset{A=4}{\longrightarrow} t = t_1 \underset{A=1,2,3}{\longrightarrow} \vec{x}_2 \longrightarrow \ldots$$

P. Delva, U. Kostić, and A. Čadež, J. Adv. Space Res. **47**, 370 (2011).

- Numerical algorithms:

allowing to draw $(t, \vec{x})$ from $\{t^A, \vec{\gamma}_A(t^A)\}$

S. Bancroft, IEEE Trans. Aerosp. Electron. Syst. **21**, 56 (1985).

L. O. Krause, ibid. **23**, 225 (1987).

- Exact covariant expression $\Longrightarrow$ $x^\alpha = f(t^A, \vec{\gamma}_A(t^A)) = \kappa^\alpha(t^A)$.

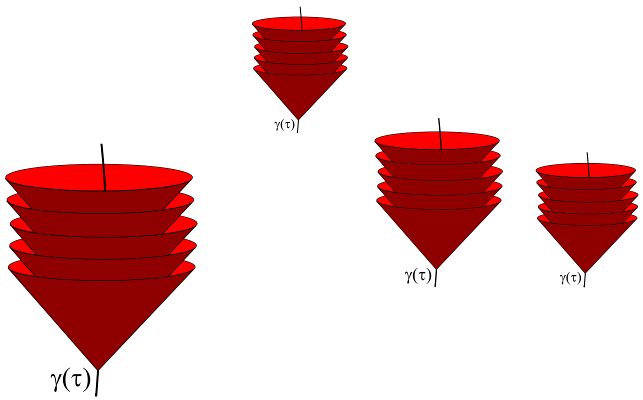
B.Coll et al. CQG **27**, 065013 (2010).

Kleusberg's algorithm (1994). See G. Strang and K. Borre, *Linear Algebra, Geodesy, and GPS* (Wellesley-Cambridge Press, 1997).

Location problem in Minkowski space-time

- Abel and Chaffee used Minkowskian algebra to state the location problem,

  - J. S. Abel, J. W. Chaffee, IEEE Trans. Aerosp. Electron. Syst. **27**, 952 (1991).
  - J. W. Chaffee, J. S. Abel, IEEE Trans. Aerosp. Electron. Syst. **30**, 1021 (1994).



- In the case of the flat space-time, an explicit form of the solution of the location problem for arbitrary emitters motions is given by

$$x = \gamma_4 + y_* - \frac{y_*^2\,\chi}{(y_* \cdot \chi) + \hat{\epsilon}\sqrt{(y_* \cdot \chi)^2 - y_*^2 \chi^2}}$$

χ y_*

$\hat{\epsilon}$

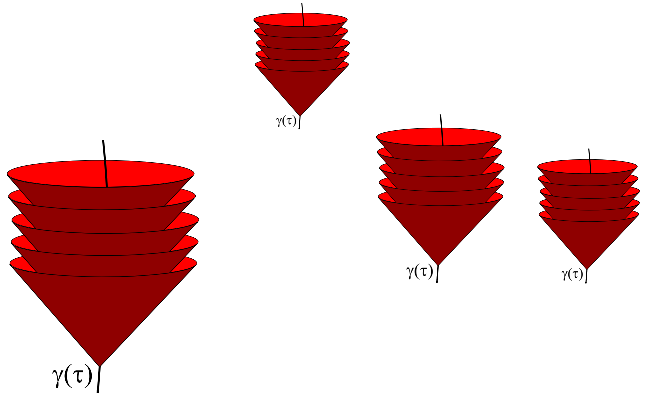
- B. Coll, et al. , Positioning systems in Minkowski spacetime: from emission to inertial coordinates, Class. Quantum Gravit. **27**, 065013 (2010).

Non-uniqueness of solutions: 'bifurcation problem'

- The non-uniqueness of the solutions in the location problem has been previously considered in connection with GPS,

  - R. O. Schmidt, A new approach to geometry range difference location, IEEE Trans. Aerospace & Electronic Systems **8**, 821(1972).
  - J. S. Abel and J. W. Chaffee, Existence and uniqueness of GPS solutions, IEEE Trans. Aerospace & Electronic Systems **27**, 952 (1991).
  - J. W. Chaffee and J. S. Abel, On the exact solutions of pseudorange equations IEEE Trans. Aerospace & Electronic Systems **30**, 1021 (1994).
  - E. W. Grafarend and J. Shan, A closed- form solution of the nonlinear pseudo-ranging equations (GPS), Artificial satellites, Planetary geodesy No 28 Special Issue on the XXX-th Anniversary of the Departament of Planetary Geodesy Vol 31 No 3 133-147 (Polish Academy of Sciences, Space Research Centre, Warszava 1996).

- Relativistic positioning system: set of four emitters A $(A = 1,2,3,4)$, of world-lines $\gamma_A(\tau^A)$, broadcasting their respective proper times τ^A by means of electromagnetic signals.



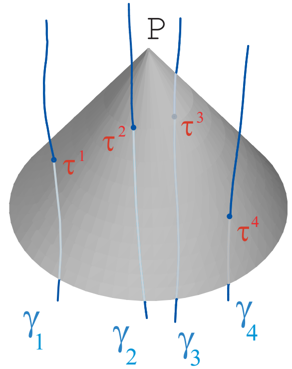
- Emission region of a RPS: set $\mathcal{R}$ of events reached by the broadcast signals.

- Emission coordinates of P: ordered 4-tuple of proper times $\{\tau^A\}$ received at P.

  - All the gradients $d\tau^A$ are light-like.

    This property determines the causal class of every relativistic emission coordinate system,

$$\{\mathrm{e\,e\,e\,e}, \mathrm{E\,E\,E\,E\,E\,E}, l\,l\,l\,l\}$$



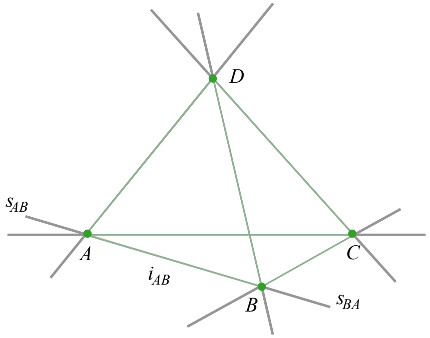
Relativistic emission coordinates with $v < 1$

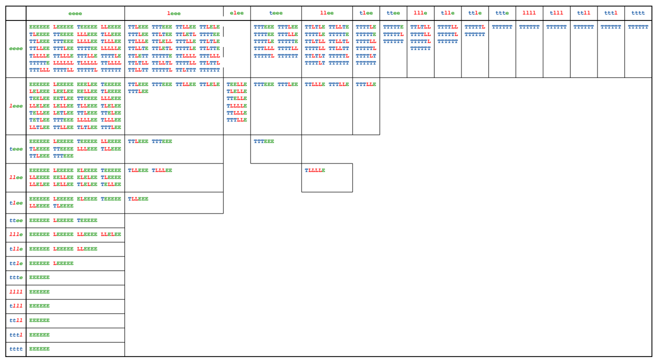
B.Coll et al., PRD **80**, 064038 (2009).

- The causal classes of the relativistic emission coordinate systems with $v < 1$ are of the form:

$$\{\mathrm{c}_1\,\mathrm{c}_2\,\mathrm{c}_3\,\mathrm{c}_4,\ \mathrm{C}_{12}\,\mathrm{C}_{13}\,\mathrm{C}_{14}\,\mathrm{C}_{23}\,\mathrm{C}_{24}\,\mathrm{C}_{34},\ e\,e\,e\,e\}$$

- Depending on the different configurations of the stationary emitters and/or of the different values of the velocity $v < 1$, the relativistic emission coordinate systems may present space-time regions of 102 different causal classes.





The 199 Causal Classes of Space –Time Frames

	eeee	leee	elee	teee	llee	tlee	ttee	llle	tlle	ttle	ttte	llll	tlll	ttll	tttl	tttt
eeee	EEEEEE LEEEEE TEEEEE LLEEEE TLEEEE TTEEEE LLLEEE TLLEEE TTLEEE TTTEEE LLLLEE TLLLEE TTLLEE TTTLEE TTTTEE LLLLLE TLLLLE TTLLLE TTTLLE TTTTLE TTTTTE LLLLLL TLLLLL TTLLLL TTTLLL TTTTLL TTTTTL TTTTTT	TTLEEE TTTEEE TTLLEE TTLELE TTTLEE TTLTEE TTLETL TTTTEE TTLLLE TTLELL TTTLLE TTLTLE TTLLTE TTLETL TTTTLE TTLTTE TTLETT TTTTTE TTLLLL TTTLLL TTLTLL TTLLTL TTTTLL TTLTTL TTLLTT TTTTTL TTLTTT TTTTTT		TTTEEE TTTLEE TTTTEE TTTLLE TTTTLE TTTTTE TTTLLL TTTTLL TTTTTL TTTTTT	TTLTLE TTLLTE TTTTLE TTTTTE TTLTLL TTLLTL TTTTLL TTLLTT TTLTLT TTTTTL TTTTLT TTTTTT	TTTTLE TTTTTE TTTTLL TTTTTL TTTTLT TTTTTT	TTTTTE TTTTTL TTTTTT	TTLTLL TTTTLL TTTTTL TTTTTT	TTTTLL TTTTTL TTTTTT	TTTTTL TTTTTT	TTTTTT	TTTTTT	TTTTTT	TTTTTT	TTTTTT	TTTTTT
leee	EEEEEE LEEEEE EEELEE TEEEEE LELEEE LEELEE EELLEE TLEEEE TEELEE EETLEE TTEEEE LLLEEE LLELEE LELLEE TLLEEE TLELEE TELLEE LETLEE TTLEEE TTELEE TETLEE TTTEEE LLLLEE TLLLEE LLTLEE TTLLEE TLTLEE TTTLEE	TTLEEE TTTEEE TTLLEE TTLELE TTTLEE	TEELLE TLELLE TTELLE TLLLLE TTLLLE TTTLLE	TTTEEE TTTLEE	TTLLLE TTTLLE	TTTLLE										
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tlee	EEEEEE LEEEEE ELEEEE TEEEEE LLEEEE TLEEEE	TLLEEE														
ttee	EEEEEE LEEEEE TEEEEE															
llle	EEEEEE LEEEEE LLEEEE LLELEE															
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tlll	EEEEEE															
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tttl	EEEEEE															
tttt	EEEEEE															

B. Coll and J. A. Morales, Int. Jour. Theor. Phys. **31**, 1045-1062 (1992).

- Standard emission data set: $E \equiv \{\gamma_A(\tau^A), \{\tau^A\}\}$
  - the emitters world-lines (referred to a specific coordinate system $\{x^\alpha\}$) and the values of the emission coordinates received by a user.
- Standard location problem:
  - to find the coordinates $\{x^\alpha\}$ of the user from the sole data E.

$$x^\alpha = f(\tau^A, \gamma_A(\tau^A)) = \kappa^\alpha(\tau^A)??$$

In a flat space-time, the standard location problem in RPS is the problem of finding (from the sole data E) the coordinate transformation $\kappa^\alpha(\tau^A)$ from emission to inertial coordinates.

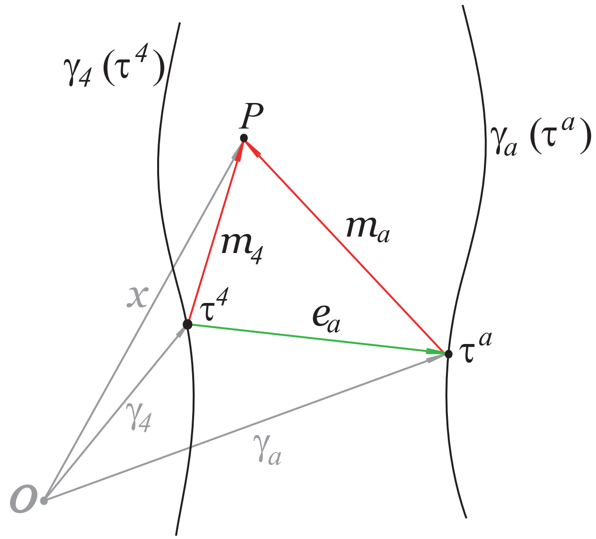
Quantities associated to the configuration of the emitters

- Configuration of the emitters for an event P: set of four events $\{\gamma_A(\tau^A)\}$ of the emitters at the emission times $\{\tau^A\}$ received at P.

  - Let us take the fourth emitter as the reference emitter,

$$e_a = m_4 - m_a = \gamma_a - \gamma_4\,, \quad (a = 1, 2, 3).$$

m_A ligth-like and future pointing



- Configuration scalars: $\Omega_a = \frac{1}{2}(e_a)^2$.

- Configuration vector: $\chi \equiv *(e_1 \wedge e_2 \wedge e_3)$.

- Configuration bivector: $H \equiv *(\Omega_1\, e_2 \wedge e_3 + \Omega_2\, e_3 \wedge e_1 + \Omega_3\, e_1 \wedge e_2)$.

All these quantities are computable from the sole standard data $\{\gamma_A(\tau^A), \{\tau^A\}\}$.

Location problem in RPS   CQG **27**, 065013 (2010)

- Location problem: to determine the inertial coordinates of a user from a standard set of data.

The coordinate transformation $x = \kappa(\tau^A)$ is given by:

$$x = \gamma_4 + y_* - \frac{y_*^2}{(y_* \cdot \chi) + \hat{\epsilon}\sqrt{\Delta}}\,\chi$$

$y_* = \dfrac{1}{\xi \cdot \chi}\, i(\xi)H$, ξ being any vector transversal to the configuration, $\xi \cdot \chi \neq 0$.

$\Delta \equiv (y_* \cdot \chi)^2 - y_*^2 \chi^2 = -\frac{1}{2} H_{\mu\nu} H^{\mu\nu} \geq 0$

- y_* and Δ are computable from the standard data $E \equiv \{\gamma_A(\tau^A), \{\tau^A\}\}$.
- $\hat{\epsilon}$ is not always computable from the sole standard data E.

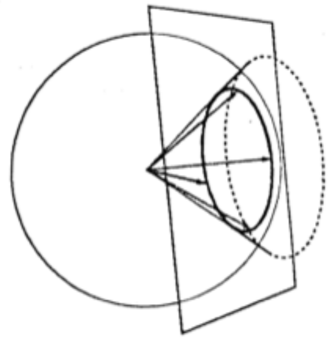
Orientation $\hat{\epsilon}$ of a RPS

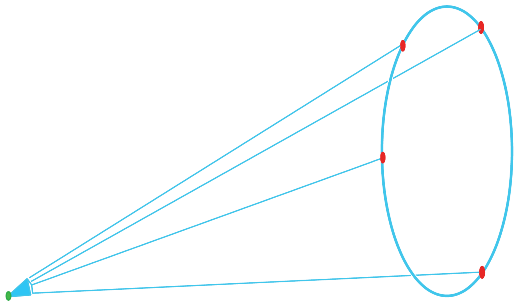
- Characteristic emission function $\Theta : \mathcal{R} \to \mathbb{R}^4$, $\Theta(x) = (\tau^A)$, Θ is not a 1-1 map.
- Orientation of a RPS at the event x: sign of the Jacobian determinant of Θ at x,

$$\hat{\epsilon} \equiv \mathit{sgn}\, j_\Theta(x) = \mathit{sgn}[*(d\tau^1 \wedge d\tau^2 \wedge d\tau^3 \wedge d\tau^4)].$$

- Zero Jacobian hypersurface: $\mathcal{J} \equiv \{x \,|\, j_\Theta(x) = 0\}$.

- Four null directions are linearly dependent iff the four space-like directions giving for any observer the relative propagation of light lie in a cone.





- Then, $\mathcal{J}$ consists in those events for which any user at them can see the four emitters on a circle on its celestial sphere.

(B. Coll, J. M. Pozo, Salamanca-2005)

Emission coordinate region of a RPS

- Emission coordinate region ($\mathcal{C} = \mathcal{R} - \mathcal{J}$): where Θ is invertible, $\mathcal{C} = \mathcal{C}_s \cup \mathcal{C}_\ell \cup \mathcal{C}_t$

$$\mathcal{C}_s \equiv \{x \in \mathcal{C} \,|\, \chi^2 < 0\}, \quad \mathcal{C}_\ell \equiv \{x \in \mathcal{C} | \, \chi^2 = 0\}, \quad \mathcal{C}_t \equiv \{x \in \mathcal{C} \,|\, \chi^2 > 0\}$$

  - From the sole standard emission data $E \equiv \{\gamma_A(\tau^A), \{\tau^A\}\}$, a user knows the configuration region ($\mathcal{C}_s$, $\mathcal{C}_\ell$, or $\mathcal{C}_t$) where he is.

- Central region of a RPS: $\mathcal{C}^C \equiv \mathcal{C}_s \cup \mathcal{C}_\ell$

  - In the central region $\mathcal{C}^C$, the orientation $\hat{\epsilon}$ of a relativistic positioning system is constant, and may be evaluated from the sole standard emission data E:

$$\forall x \in \mathcal{C}^C, \qquad \hat{\epsilon} = \mathit{sgn}\,(u \cdot \chi)$$

    where u is any future pointing time-like vector.

- At any event of the emission coordinate region $\mathcal{C}$, the orientation $\hat{\epsilon}$ can be obtained from the relative positions of the emitters on the celestial sphere of the user at this event.

Observational rule to determine $\hat{\epsilon}$ at the user position

At any event of the emission coordinate region $\mathcal{C}$, the orientation $\hat{\epsilon}$ can be obtained from the relative positions of the emitters on the celestial sphere of the user at this event.

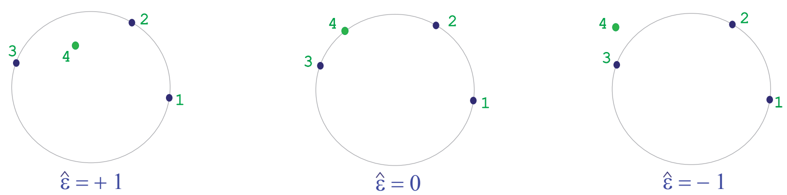
Denote by $\vec{n}_A$ the unit vectors giving the relative directions of propagation of the signals.

- The orientation $\hat{\epsilon}$ of a relativistic positioning system is given by

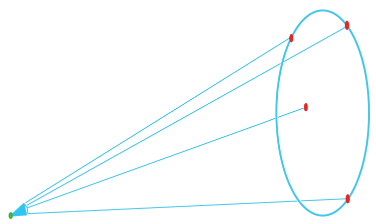
$$\hat{\epsilon} = sgn\,[(1 - \vec{n}_4 \cdot \vec{L})(\vec{n}_1, \vec{n}_2, \vec{n}_3)]$$

where $\vec{L} \equiv \vec{L}^1 + \vec{L}^2 + \vec{L}^3$, with $\vec{L}^a = \dfrac{\epsilon^{abc}\vec{n}_b \times \vec{n}_c}{2(\vec{n}_1, \vec{n}_2, \vec{n}_3)}, \quad (a, b, c = 1, 2, 3).$

- Consider the oriented half-cone containing $\vec{n}_1$, $\vec{n}_2$ and $\vec{n}_3$.
  If $\vec{n}_4$ is in its interior, $\hat{\epsilon} = -sgn[(\vec{n}_1, \vec{n}_2, \vec{n}_3)]$.
  Otherwise, $\hat{\epsilon} = sgn[(\vec{n}_1, \vec{n}_2, \vec{n}_3)]$.



Application of the rule for $(\vec{n}_1, \vec{n}_2, \vec{n}_3) < 0$.



- The relative positions of the emitters in the celestial sphere of a user are Lorentz invariant.

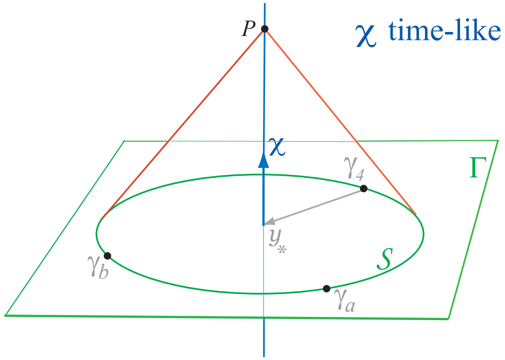
Location problem in RPS

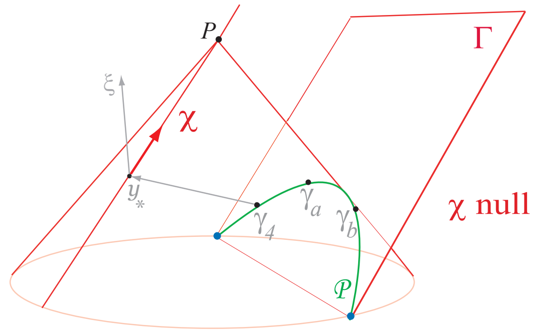
$P \in \mathcal{C}_s \cup \mathcal{C}_\ell$ No bifurcation
(central region)

$$\chi^2 \leq 0$$

$$\hat{\epsilon} = sgn(u \cdot \chi)$$

$\forall u$ future pointing time-like vector





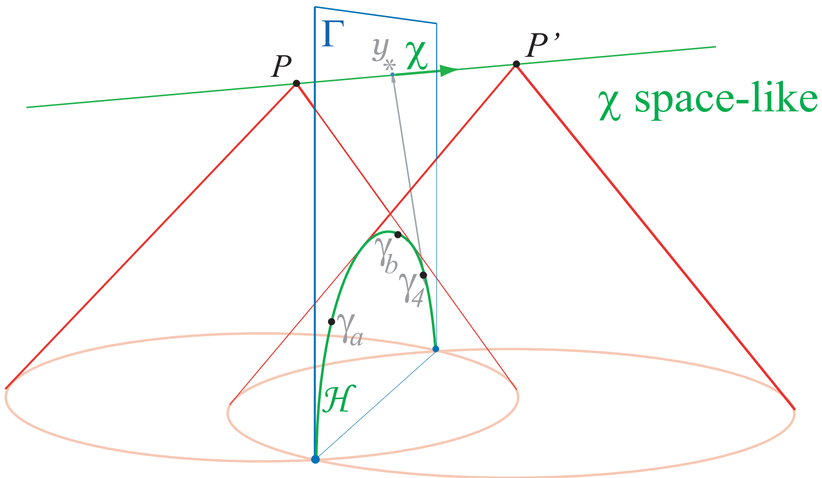
* *

$P, P' \in \mathcal{C}_t$ Bifurcation
(time-like coordinate region)

$$\chi^2 > 0$$

$\Theta(P) = \Theta(P')$ 'conjugate events'

Observational rule $\rightarrow$ $\hat{\epsilon}$ at the user position



Emission coordinate domains

- The emission coordinate region contains two coordinate domains:

$$\mathcal{C} = \mathcal{C}_s \cup \mathcal{C}_\ell \cup \mathcal{C}_t = \mathcal{C}^F \cup \mathcal{C}^B$$

which are called the front ($\mathcal{C}^F$) and the back ($\mathcal{C}^B$) emission coordinate domains.

- The border between these domains is the hypersurface $\mathcal{J} = \{x \mid j_\Theta(x) = 0\}$.

Timelike coordinate region:

$$\mathcal{C}_t = \mathcal{C}_t^F \cup \mathcal{C}^B, \quad \Theta(\mathcal{C}_t^F) = \Theta(\mathcal{C}^B)$$

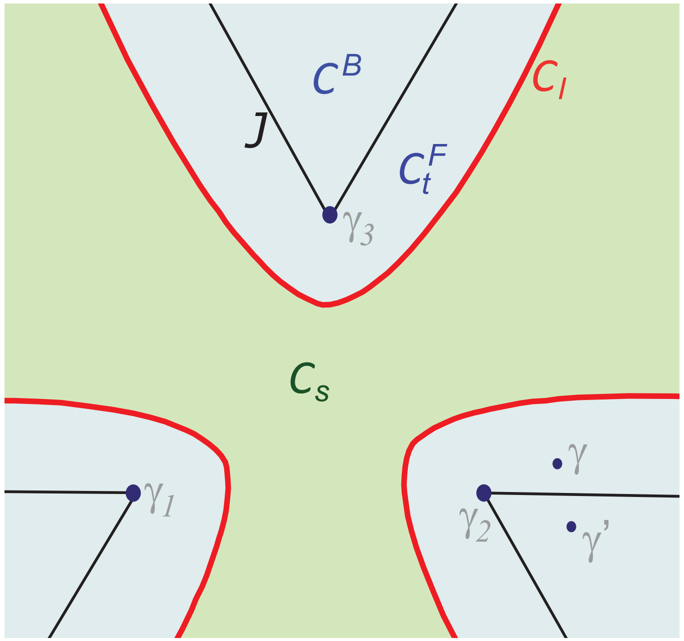
Back emission coordinate domain:

$$\mathcal{C}^B = \mathcal{C}_t - \mathcal{C}_t^F$$

Front emission coordinate domain:

$$\mathcal{C}^F = \mathcal{C}_s \cup \mathcal{C}_\ell \cup \mathcal{C}_t^F$$

Central region: $\mathcal{C}^C = \mathcal{C}_s \cup \mathcal{C}_\ell$



Symmetric and stationary RPS in 3D

Location problem: relative formulation

- In practice, the location problem is formulated with respect to an (inertial) observer u, $u^2 = -1$. The unknown space-time position x of the user can be written as:

$$x = x^0 u + \vec{x}, \quad x^0 = -x \cdot u, \quad \vec{x} \cdot u = 0,$$

  $\{x^0, \vec{x}\}$ being the inertial components of x.

- Relatively to u, the standard location problem consists in finding the coordinate transformation from emission to inertial coordinates,

$$x^0(\tau^A) \quad \text{and} \quad \vec{x}(\tau^A)$$

  when the motions of the emitters are known in the inertial coordinate system.

- The position vectors γ_A of the emitters at the emission times split as,

$$\gamma_A = t_A u + \vec{\gamma}_A, \qquad (A = 1, 2, 3, 4)$$

  where $t_A \equiv \gamma_A^0$ is the inertial time of the event $\gamma_A(\tau^A)$ measured by u.

Splitting of the configuration vector χ

$$\chi = (\vec{e}_1, \vec{e}_2, \vec{e}_3)\, u + \sigma_1\, \vec{e}_2 \times \vec{e}_3 + \sigma_2\, \vec{e}_3 \times \vec{e}_1 + \sigma_3\, \vec{e}_1 \times \vec{e}_2$$

- Relatively to u, the configuration vector is expressed as $\chi = \chi^0 u + \vec{\chi}$ with

$$\chi^0 = (\vec{e}_1, \vec{e}_2, \vec{e}_3), \qquad \vec{\chi} = \frac{1}{2}\epsilon^{abc}\sigma_a\, \vec{e}_b \times \vec{e}_c$$

where $\{\sigma_a, \vec{e}_a\}$ are the relative components of the position vectors of the referred emitters,

$$e_a = \gamma_a - \gamma_4 = \sigma_a u + \vec{e}_a \qquad (a = 1, 2, 3)$$

with $\sigma_a = t_a - t_4$ and $\vec{e}_a = \vec{\gamma}_a - \vec{\gamma}_4$.

  - $|\chi^0|$: volume of the parallelepiped defined by the relative positions $\vec{e}_a$ of the referred emitters.
  - $\vec{\chi}$: weighted vector-area.

Splitting of the configuration bivector H

$$\vec{S} = \Omega_1\, \vec{e}_2 \times \vec{e}_3 + \Omega_2\, \vec{e}_3 \times \vec{e}_1 + \Omega_3\, \vec{e}_1 \times \vec{e}_2$$

$$\vec{B} = \Omega_1(-\sigma_2\vec{e}_3 + \sigma_3\vec{e}_2) + \Omega_2(-\sigma_3\vec{e}_1 + \sigma_1\vec{e}_3) + \Omega_3(-\sigma_1\vec{e}_2 + \sigma_2\vec{e}_1)$$

- The configuration bivector can be written as

$$H = u \wedge \vec{S} - *(u \wedge \vec{B}), \qquad \vec{S} \equiv -i(u)H, \quad \vec{B} \equiv -i(u) * H,$$

with $\vec{S}$ and $\vec{B}$ as the electric and magnetic part of H relative to u.

- Algebraic invariants of H:
  - $\Delta = -\frac{1}{2} H_{\mu\nu} H^{\mu\nu} = \vec{S}^2 - \vec{B}^2 \geq 0, \quad sgn\,(\Delta) = sgn\,[(\chi \cdot m_4)^2]$.
  - $H_{\mu\nu}(*H)^{\mu\nu} = 0 \quad \Longrightarrow \quad \vec{S} \cdot \vec{B} = 0.$

- For an event $x \in \mathcal{R}$,

$$j_\Theta(x) = 0 \iff \Delta = 0 \iff \vec{S}^2 = \vec{B}^2 \iff H \text{ is a null bivector.}$$

$\Longrightarrow$ On the border $\mathcal{J}$, the location of a user may be unambiguously solved.

Splitting of the particular solution y_*

$$y_* = \tfrac{1}{D}\,(\vec{\xi}\cdot\vec{S}\;u + \vec{S} + \vec{\xi}\times\vec{B})$$

- The transversal vector ξ can be always chosen so that $\xi^0 = 1$, $\xi = u + \vec{\xi}$, so that the transversality condition, $\xi\cdot\chi \neq 0$, is expressed as

$$D \equiv (\vec{e}_1, \vec{e}_2, \vec{e}_3) - \frac{1}{2}\epsilon^{abc}\sigma_a\,(\vec{\xi}, \vec{e}_b, \vec{e}_c) \neq 0.$$

- Relatively to an inertial observer u, the particular solution y_* orthogonal to $\xi = u + \vec{\xi}$ is given by

$$y_* = \frac{1}{D}\,(\vec{\xi}\cdot\vec{S}\;u + \vec{S} + \vec{\xi}\times\vec{B})$$

where $\vec{S}$ and $\vec{B}$ are,respectively, the electric and magnetic parts of H (that are computed from $\vec{e}_a$).

Splitting of the solution $x = \gamma_4 + y_* + \lambda\chi$

$$\lambda = \frac{-(y_*^0)^2 + \vec{y}_*{}^2}{(-y_*^0\chi^0 + \vec{y}_* \cdot \vec{\chi}) + \hat{\epsilon}\sqrt{\vec{S}^2 - \vec{B}^2}}$$

and obtaining the orientation $\hat{\epsilon}$

- When $\chi^2 \leq 0$ there is a sole emission solution x. 'No bifurcation'

  To obtain it, take $\hat{\epsilon} = sgn(u \cdot \chi)$ (where u is any future pointing time-like vector).

  In particular, if u is an inertial observer,

  $$\hat{\epsilon} = -sgn(\chi^0) = -sgn[(\vec{e}_1, \vec{e}_2, \vec{e}_3)],$$

- When $\chi^2 > 0$ there are two emission solutions, x and x' (conjugate events).

  They only differ by their orientation $\hat{\epsilon}$.

  The observational method allows us to determine $\hat{\epsilon}$, and then to obtain the solution corresponding to the real user (solving bifurcation).

Summary and ending comments

- The standard emission data set $\{\gamma_A(\tau^A), \{\tau^A\}\}$ is generically insufficient to locate a user of a positioning system in an inertial system.
  - The bifurcation problem appears in the time-like region, $\mathcal{C}_t$, ($\chi^2 > 0$).
  - The knowledge of the orientation in addition to the standard emission data set $\{\gamma_A(\tau^A), \{\tau^A\}\}$ solves completely the bifurcation problem.
- In current practical situations in present-day GNSS, the bifurcation problem may be solved by hand. If a user stays near the Earth surface the right solution is the nearest to the Earth radius.
  - However, in extended GNSS or more general positioning systems in the Solar System, the bifurcation problem cannot be so easily avoided.
- Among all the available 4-tuples of satellites of a GPS or Galileo constellation, the more appropriate one should minimize positioning errors avoiding bifurcation (next talk by D. Sáez and N. Puchades).