RELATIVISTIC POSITIONING SYSTEMS AND GRAVITATIONAL PERTURBATIONS

A. Gomboc, M. Horvat, U. Kostić (UL)
and
S. Carloni (ESA), P. Delva (Obs. Paris)

ESA PECS project

- ESA PECS Relativistic Global Navigation System
- 2011-2014
- continuation of the ESA ACT Ariadna projects (A. Čadež, U. Kostić, P. Delva)
- refine the description of the system
- same concept (ABC, recovery of constants of motion, refining the Hamiltonian)

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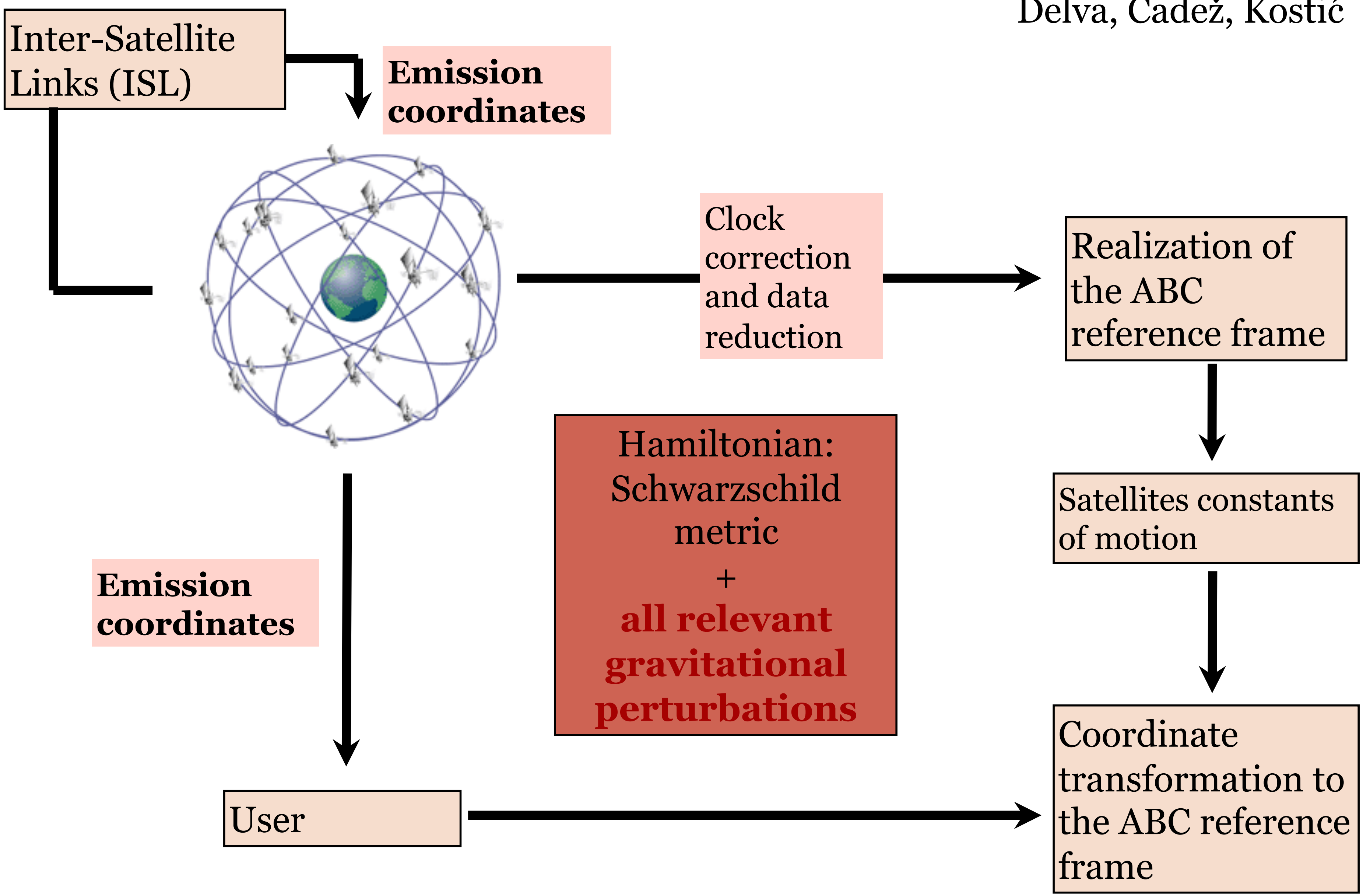
gravitational perturbations

Delva, Čadež, Kostić



Project goals

1. add first order gravitational perturbations to the Schwarzschild metric
    - find perturbation coefficients describing all known gravitational perturbations: due to the Earth's multipoles, tides, rotation; gravity of the Moon, the Sun, and planets (Venus, Jupiter)

2. solve the perturbed geodesic equations
    - use Hamiltonian formalism → perturbation theory → obtain time evolution of 0^{th} order constants of motion
    - (Ariadna study: analytic solutions of 0^{th} order)

3. find accurate constants of motion
   - using inter-satellite distances measured over many periods
   - stability and degeneracies
   - (Ariadna study: done for 0^{th} order)

4. refine values of gravitational perturbation coefficients
   - use residual errors between orbit prediction and orbit determination through inter-satellite communication
   - accuracy of position
   - possible scientific applications

Perturbations

gravitational perturbations:

- Earth's multipoles
- the Moon
- the Sun
- Earth's tides
- planets (Venus, Jupiter)
- Earth's rotation

non-gravitational effects:

- Solar radiation pressure
- Earth's albedo

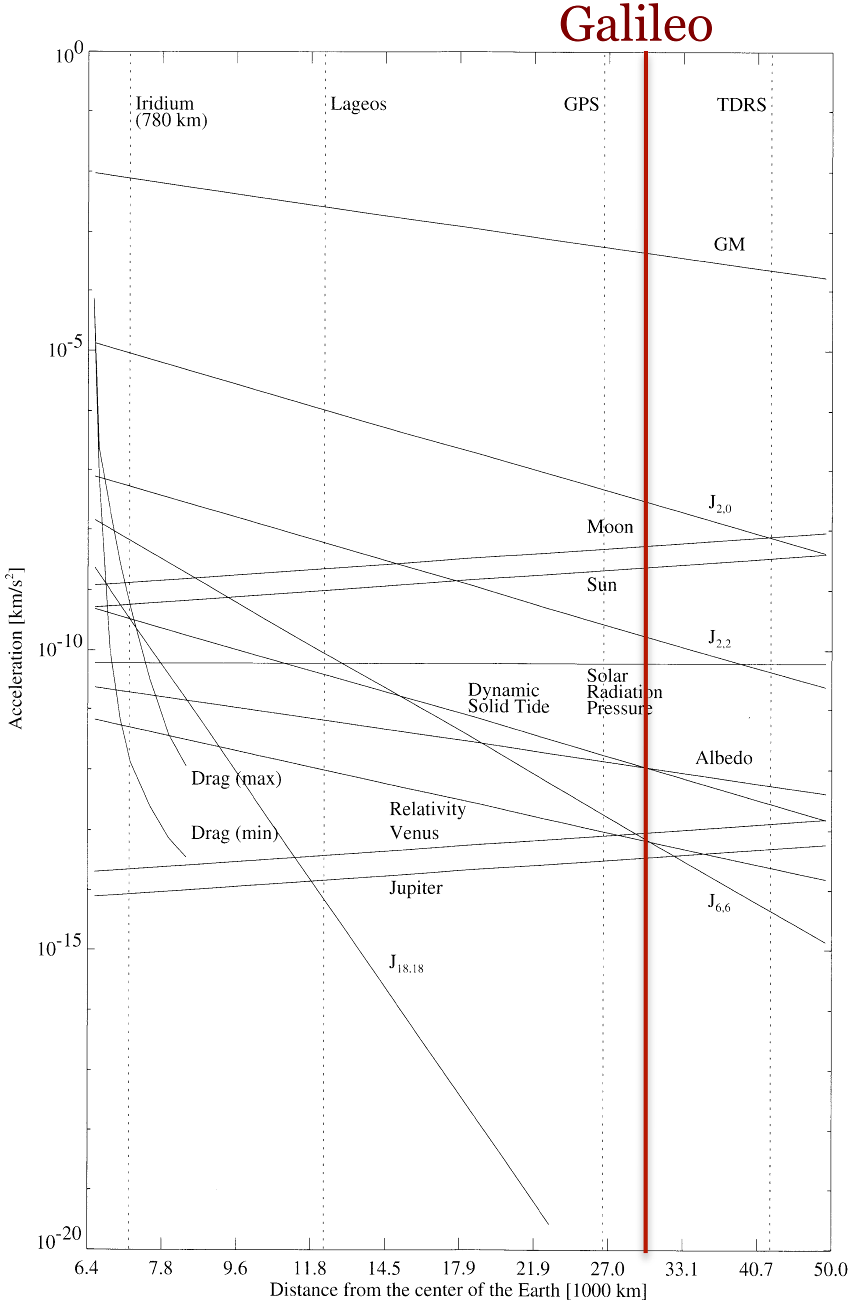


Fig. 3.1. Order of magnitude of various perturbations of a satellite orbit. See text for further explanations.

Montenbruck & Gill, 2005

Perturbations in Schwarzschild background

- Schwarzschild background (spherically symmetric, time independent): $g^{(0)}_{\mu\nu}$
- linear perturbation theory
- perturbations: $h_{\mu\nu} \ll g^{(0)}_{\mu\nu}$
- perturbed metric: $g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu}$
- Einstein's equation:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu}$$

General equations:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -8\pi T_{\mu\nu}$$

$$T_{\mu\nu} = T^{(0)}_{\mu\nu} + \delta T_{\mu\nu} \qquad R^{(0)}_{\mu\nu} - \frac{1}{2}g^{(0)}_{\mu\nu}R^{(0)} = -8\pi T^{(0)}_{\mu\nu}$$

$$R_{\mu\nu} = R^{(0)}_{\mu\nu} + \delta R_{\mu\nu} \qquad R^{(0)}_{\mu\nu} - \frac{1}{2}g^{(0)}_{\mu\nu}R^{(0)} = 0 \quad \Longrightarrow \quad R^{(0)} = 0\ ,\ R^{(0)}_{\mu\nu} = 0$$

$$h_{\alpha}{}^{\alpha}{}_{;\mu\nu} - h_{\mu}{}^{\alpha}{}_{;\nu\alpha} - h_{\nu}{}^{\alpha}{}_{;\mu\alpha} + h_{\mu\nu;}{}^{\alpha}{}_{\alpha} + g^{(0)}_{\mu\nu}(h_{\alpha}{}^{\lambda}{}_{;}{}^{\alpha}{}_{\lambda} - h_{\lambda}{}^{\lambda}{}_{;\alpha}{}^{\alpha}) + g^{(0)}_{\mu\nu}h_{\lambda\sigma}R^{(0)\lambda\sigma} - h_{\mu\nu}R^{(0)} = -16\pi\delta T_{\mu\nu}$$

Equation A

Regge & Wheeler (1957) approach

- spherical harmonics expansion:

$$h_{\mu\nu} = \sum_{l=2}^{\infty} \sum_{m=-l}^{l} (h_{\mu\nu}^{lm})^{(o)} + (h_{\mu\nu}^{lm})^{(e)}$$

$(h_{\mu\nu}^{lm})^{(o)}$... odd parity

$(h_{\mu\nu}^{lm})^{(e)}$... even parity

- solution of Equation A:
- for odd: 3 functions of r
- for even: 7 functions of r

gauge transformations

Regge & Wheeler (RW)(1957)

$$x'^{\nu} = x^{\nu} + \xi^{\nu}$$

$$h'_{\mu\nu} = h_{\mu\nu} + \xi_{\mu;\nu} + \xi_{\nu;\mu}$$

- odd:

$$\xi^0 = \xi^1 = 0$$

$$\xi^d = \Lambda(t,r)\epsilon^{cd}Y^{lm}{}_{,d} \quad \text{for} \quad d = 2,3$$

- even:

$$\xi^0 = M_0(t,r)Y^{lm}(\theta,\phi)$$

$$\xi^1 = M_1(t,r)Y^{lm}(\theta,\phi)$$

$$\xi^2 = M_2(t,r)\partial_\theta Y^{lm}(\theta,\phi)$$

$$\xi^3 = M_2(t,r)\csc^2\theta\,\partial_\phi Y^{lm}(\theta,\phi)$$

$$(h^{lm}_{\mu\nu})^{(o)} = \left[\begin{array}{cc|cc} 0 & 0 & -h_0^{lm}\csc\theta\partial_\varphi & h_0^{lm}\sin\theta\partial_\theta \\ 0 & 0 & -h_1^{lm}\csc\theta\partial_\varphi & h_1^{lm}\sin\theta\partial_\theta \\ \hline \star & \star & 0 & 0 \\ \star & \star & 0 & 0 \end{array}\right] Y^{lm}(\theta,\varphi)$$

$$(h^{lm}_{\mu\nu})^{(e)} = \left[\begin{array}{cc|cc} H_0^{lm}(1-\frac{r_s}{r}) & H_1^{lm} & 0 & 0 \\ \star & H_2^{lm}(1-\frac{r_s}{r})^{-1} & 0 & 0 \\ \hline 0 & 0 & r^2K^{lm} & 0 \\ 0 & 0 & 0 & r^2K^{lm}\sin^2\theta \end{array}\right] Y^{lm}(\theta,\varphi)$$

- odd f.: 2, even f.: 4
- vacuum, time independent: $H_0^{lm} = H_2^{lm} = H^{lm}$
  (RW 1957, Zerilli 1970) $H_1^{lm} = 0$
  $h_1^{lm} = 0$

r_s.... Schwarzschild radius

even parity functions

we get solutions for $r>r_s$:

$$H^{lm}(r) = B_{lm}\frac{P_l(r_s/r)}{r^l(r-r_s)}$$

$$K^{lm}(r) = B_{lm}\frac{R_l(r_s/r)}{r^{l-1}(r-r_s)^2}$$

where:

$$P_l(y) = \sum^{\infty} p_i y^i = 1 + \frac{1}{2}(l-1)y + \frac{(l+2)l(l-1)}{4(2l+3)}y^2 + \frac{(l+3)(l+1)l(l-1)}{24(2l+3)}y^3 + O(y^4)$$

$$R_l(y) = \sum_{i=0}^{\infty} r_i y^i$$

odd parity functions

RW (1957), Zerilli (1970)

- vacuum, time independent: $h_1^{lm} = 0$

$$\boxed{h_0^{lm}(r) = A_{lm} r^{-l} S_l(r_s/r)}$$

where $S_l(y) = \sum_{i=0}^{\infty} s_i y^i$

- purely relativistic

Earth's multipoles – time independent

- Newtionian gravitational potential:

$$\Phi_{\rm N} = \sum_{lm} \frac{N_{lm}}{r^{l+1}} Y^{lm}$$

- perturbation:

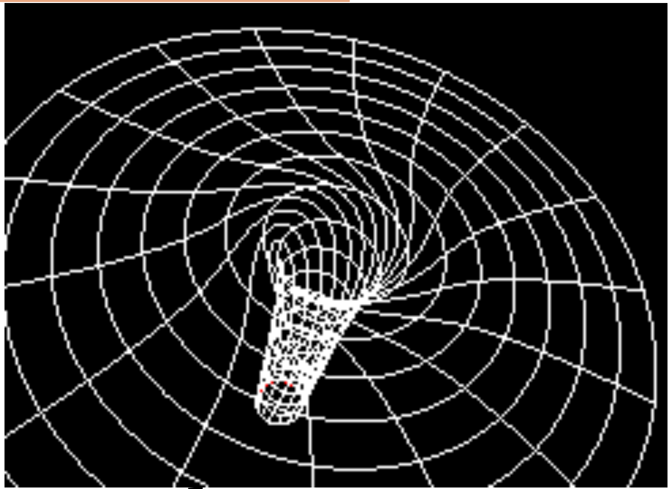
$$g_{00} = c^2 \left(1 - \frac{r_s}{r}\right) \left(1 + \frac{1}{c^2} \sum_{lm} H_0^{lm}(r) Y^{lm}(\theta, \phi)\right)$$

- lim c ⟶ ∞:

$$\boxed{B_{lm} = 2N_{lm}}$$

connection between Newtonian multipole momenta and expansion coefficients

Earth rotation



3 effects:

1. Kerr effect due to rotating monopole, to 1st order: Allison (1989)

$$h_{t\varphi}^{Kerr} = h_{\varphi t}^{Kerr} = -\frac{r_s}{r} a \sin^2\theta \quad \text{where} \quad a = \frac{\Gamma}{M_\oplus c}$$

- or with RW, odd parity function l=1, m=0:

$$A_{10} = a r_s \sqrt{\frac{4\pi}{3}} \longrightarrow h_0^{10}(r) = \sqrt{\frac{4\pi}{3}} a \frac{r_s}{r} S_1(r_s/r)$$

2. Kerr effect due to rotating (higher) multipoles – negligible in 1st order (Hartle 1967)

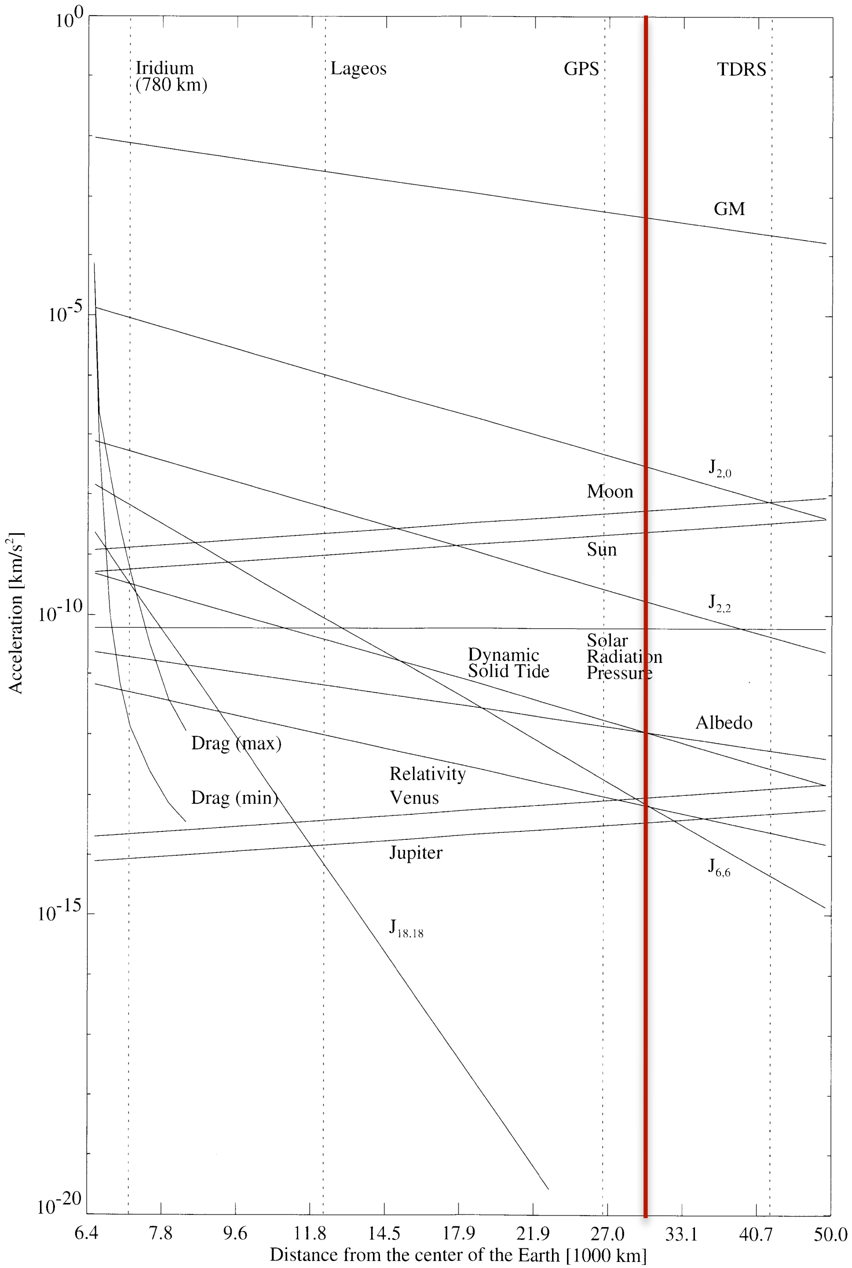


Fig. 3.1. Order of magnitude of various perturbations of a satellite orbit. See text for further explanations.

Montenbruck & Gill, 2005

3. time dependent multipoles

- perturbations vary with (small) ω:

$$\Phi_{\mathrm{N}} = \frac{GM_{\oplus}}{r} + \Delta\Phi_{\mathrm{N}} \qquad \Delta\Phi_{\mathrm{N}}(t, r, \theta, \varphi) = \sum_{lm} \frac{N_{lm}}{r^{l+1}} Y^{lm}(\theta, \varphi) e^{\pm \mathrm{i}m\omega t}$$

$$H^{lm}(T, r) = e^{\mathrm{i}mkT} \tilde{H}^{lm}(r) \,, \qquad T = ct$$

$$K^{lm}(T, r) = e^{\mathrm{i}mkT} \tilde{K}^{lm}(r) \,,$$

$$H_1^{lm}(T, r) = e^{\mathrm{i}mkT} \tilde{H}_1^{lm}(r) \,,$$

- RW (1957), Zerilli (1970): $H_1^{lm} \neq 0$

- series in frequency:

$$\tilde{H}^{lm} = \sum_{i=0}^{\infty} k^{2i} \tilde{H}^{lm(i)} , \quad \tilde{K}^{lm} = \sum_{i=0}^{\infty} k^{2i} \tilde{K}^{lm(i)} , \quad \tilde{H}_1^{lm} = \sum_{i=0}^{\infty} k^{2i+1} \tilde{H}_1^{lm(i)}$$

- to 1st order
- even:

$$\tilde{H}_1^{lm(0)} = -\frac{r^2}{l(l+1)} \left[\frac{6}{r} + \frac{1}{r_s}(l-1)(l+2)(H^{lm} + K^{lm}) - \frac{4}{r(1-r_s/r)} K^{lm} \right]$$

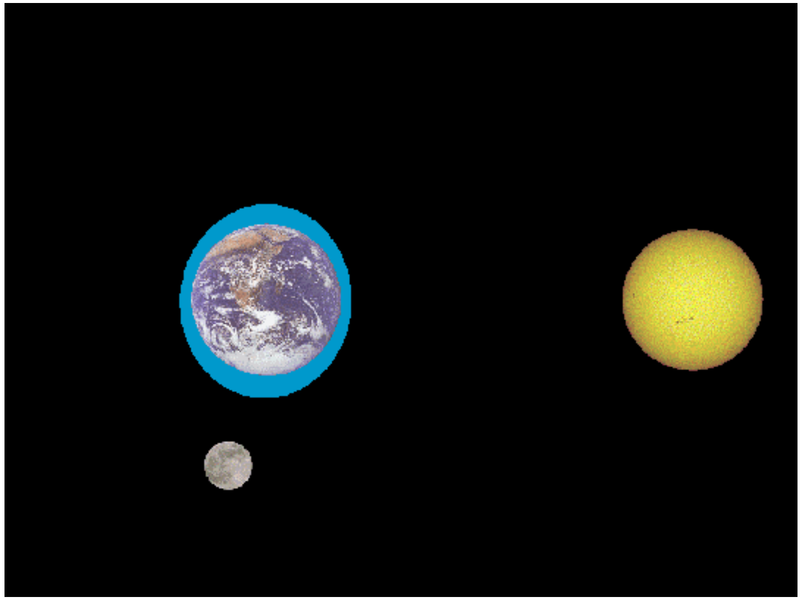
$$\tilde{H}^{lm} = \tilde{H}^{lm(0)} \qquad \tilde{K}^{lm} = \tilde{K}^{lm(0)}$$

- odd:

  solutions not flat at r $\longrightarrow \infty$

Earth tides

- solid and ocean tides
- time dependent Earth's gravitational potential
- similar to time dependent multipoles: different frequency and phase, amplitude
- literature with data on tides



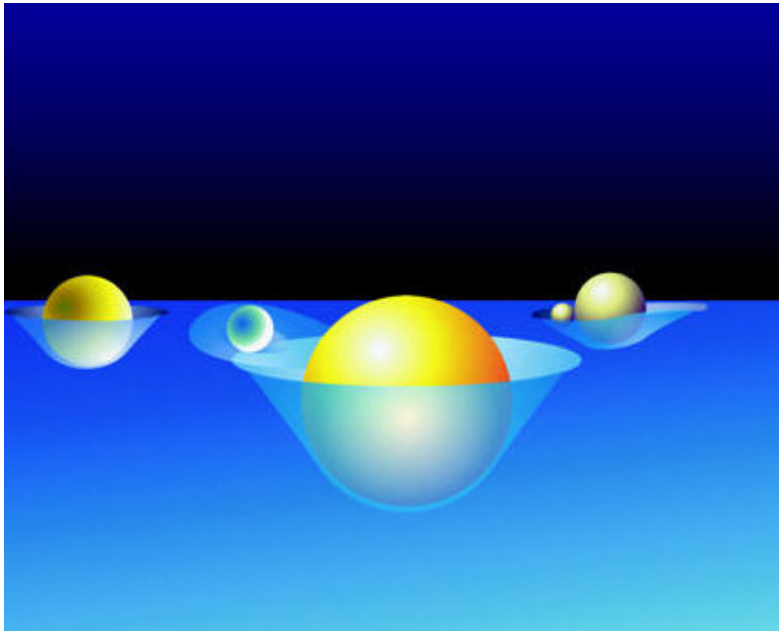
Celestial bodies

Moon, Sun, planets

- post Newtonian description – PPN: Einstein, Infeld, Hoffmann (1938), Brumberg (2007), Landau&Lifshitz (1980)

OR

- expansion – similar as multipoles



- for each celestial body:

$$U = \sum_i \frac{GM_i}{\|\mathbf{r}_i - \mathbf{r}\|} \qquad [h_{\mu\nu}] = \frac{2U}{c^2}\mathrm{diag}(-1, 1, 0, 0)$$

$$U = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} r^l M_{lm}^{\mathrm{cel}} Y^{lm}(\theta, \varphi)$$

$$M_{lm}^{\mathrm{cel}} = \sum_i \frac{4\pi G M_i}{(2l+1) r_i^{l+1}} Y^{lm*}(\theta_i, \varphi_i)$$

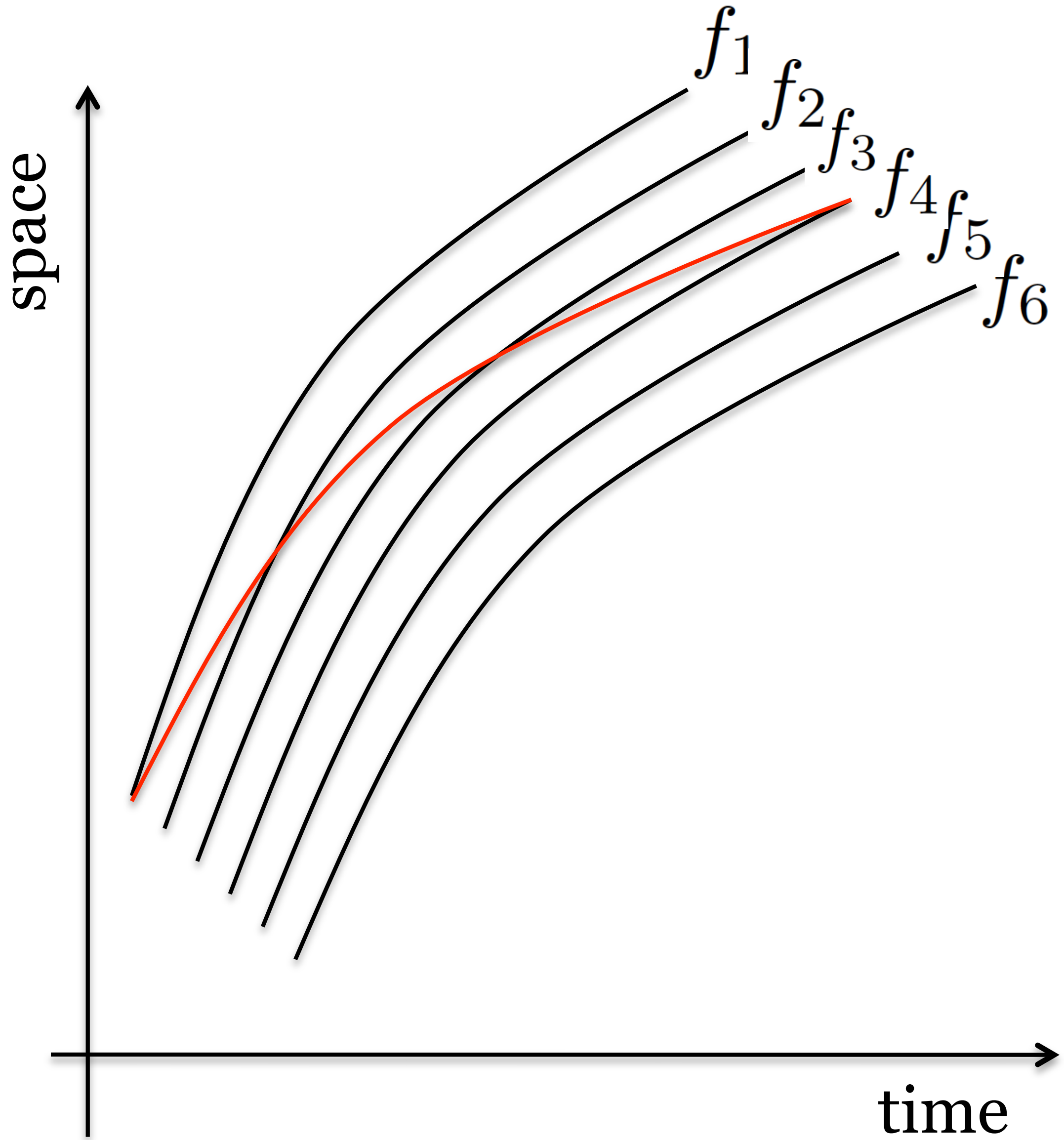
Hamiltonian formalism

$$h_{\mu\nu} = h_{\mu\nu}^{\text{multipoles}} + h_{\mu\nu}^{\text{tides}} + h_{\mu\nu}^{\text{cel}} + h_{\mu\nu}^{\text{Kerr}}$$

$$\mathcal{H} = \underbrace{\frac{1}{2} g^{\mu\nu(0)} p_\mu p_\nu}_{\mathcal{H}^{(0)}} - \underbrace{\frac{1}{2} h^{\mu\nu} p_{\mu\nu}}_{\Delta\mathcal{H}}$$

- 1^{st} order correction to constants of motion

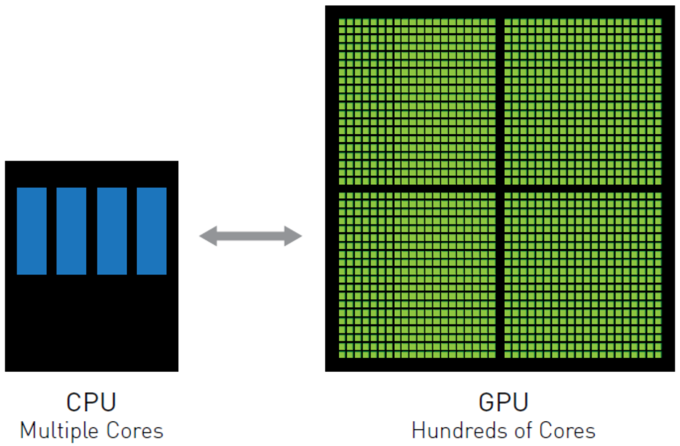
$$\dot{f} = \frac{\mathrm{d}f}{\mathrm{d}\lambda} = \frac{\partial f}{\partial \lambda} + \{f, \mathcal{H}\}$$



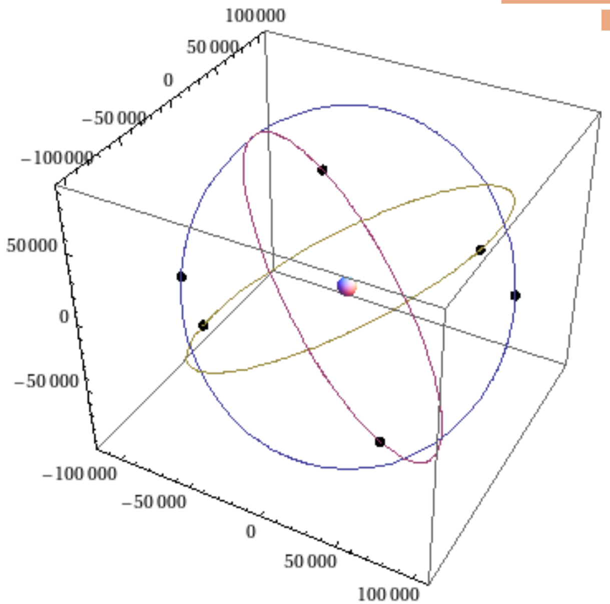
- for Kerr effect: Alisson (1989) equations for $\dot{f}$
- for others: in Mathematica

Software/Hardware

- Time-consuming minimization procedures
  - Ariadna study: 1 – 10 minutes for 0th order
  - add perturbations → even more time-consuming
- Parallelization
  - mostly independent pairs of satellites → ideal for parallelization
  - GPUs - may run few 10 – 100 times faster than a single CPU



Next steps



simulate satellite orbits in Schwarzschild background with 'gravitationally perturbed' time dependent 'constants of motion'

→

let satellites communicate – refine their constants of motions

→

use residual satellites and errors to refine the Hamiltonian, i.e. obtain improved parameters of perturbations

- Model the Galileo GNSS directly in general relativity including gravitational perturbations
- To what level can this new approach improve the accuracy and stability of the Galileo GNSS reference frame
- Scientific utilization of the GNSS, e.g. relativistic gravimetry, geology...

comments and suggestions?

Thank you!