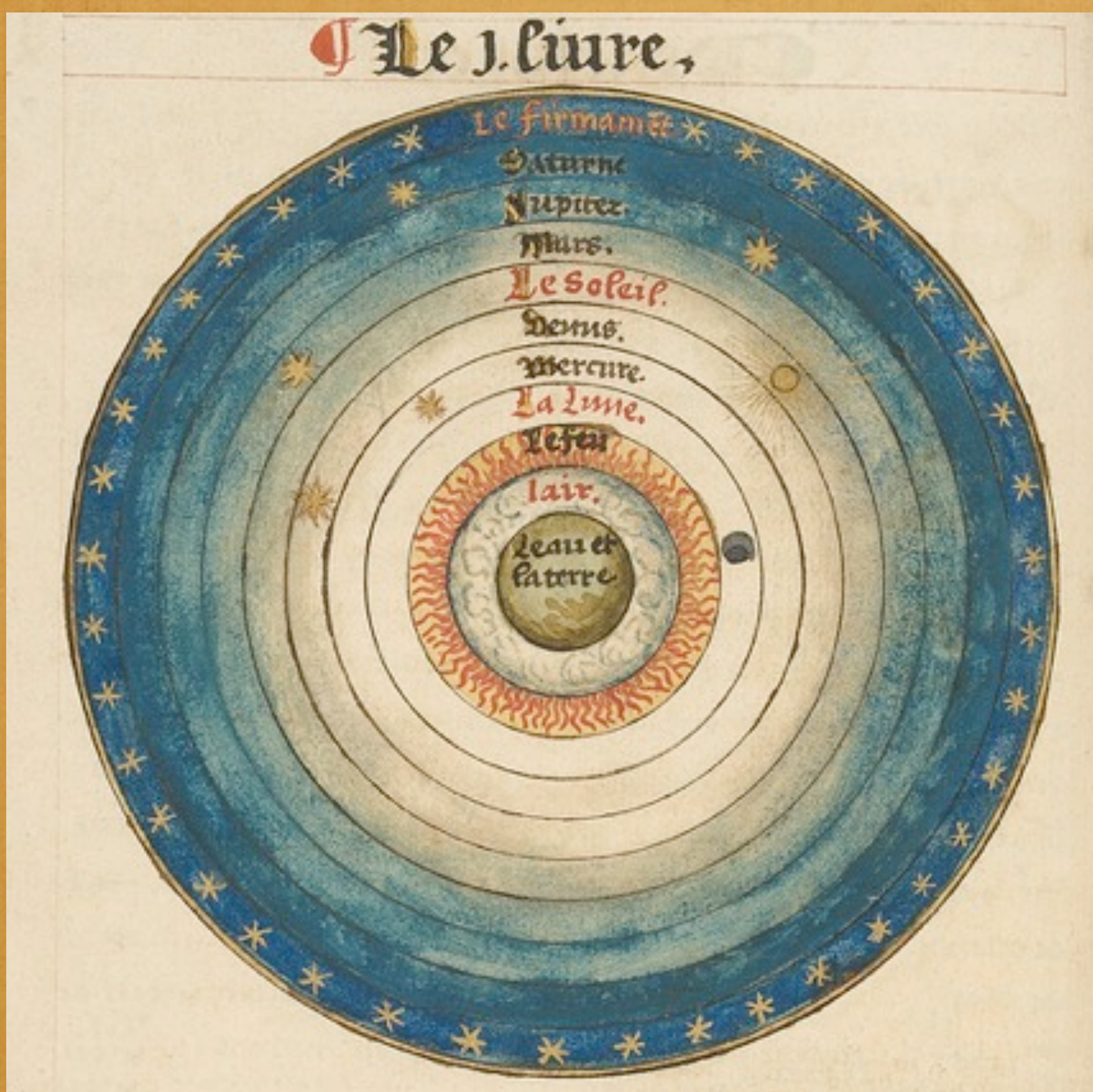
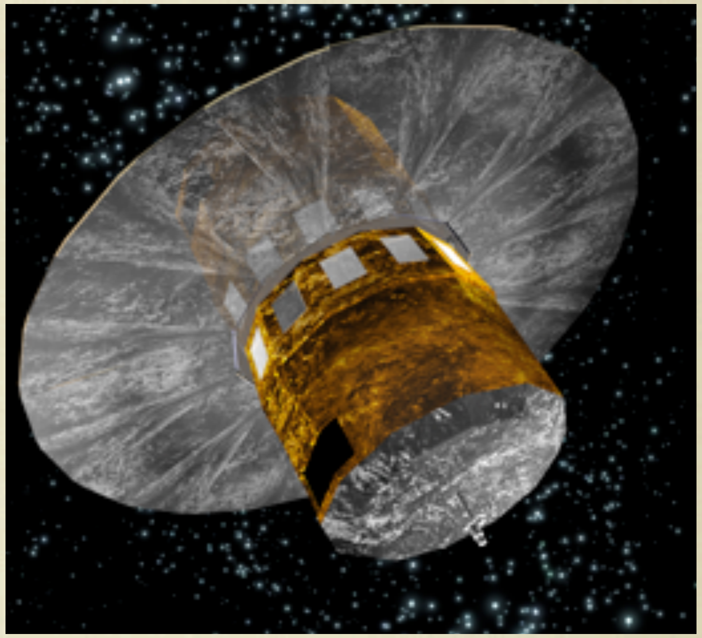
Lie perturbative analysis of the restricted two-body problem in PN-gravitation



Sante Carloni
ACT-ESA
20/09/2012

Space and RCM



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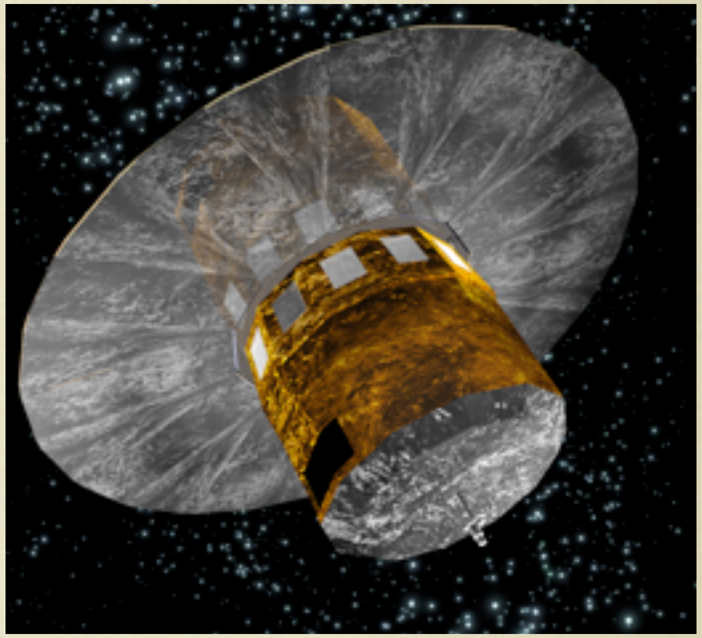

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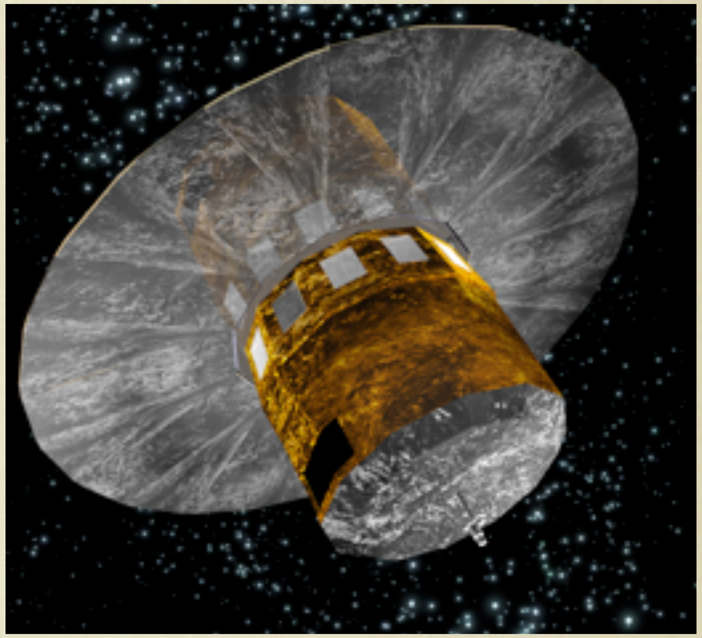

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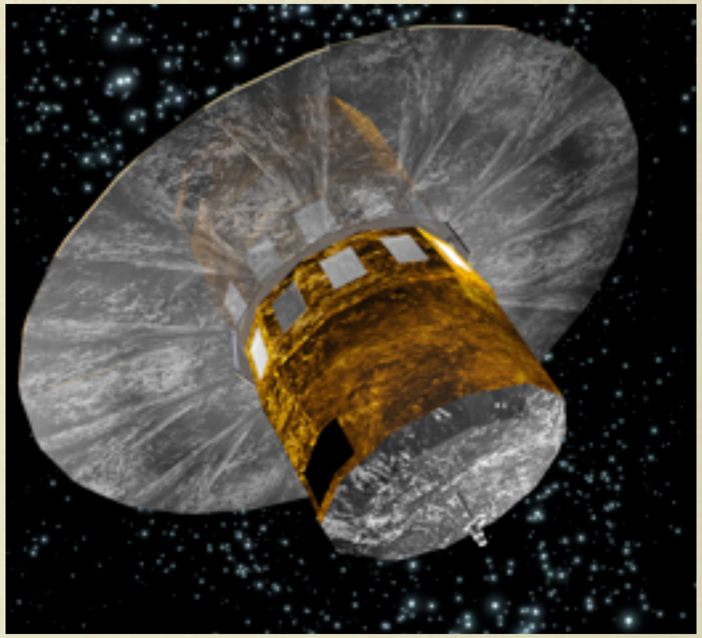

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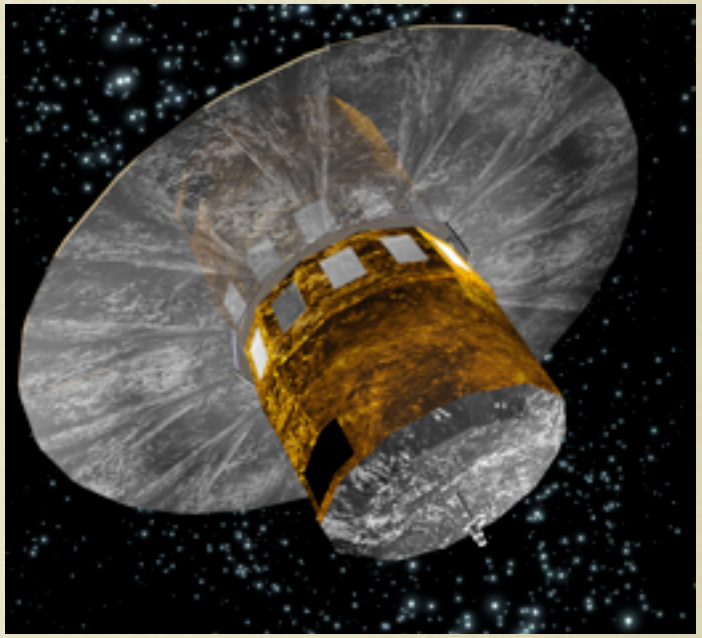
- RCM was used so far mainly to determine the gravitational waveform emitted by astrophysical objects, but its results can be used also in other ways...



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- Such calculation in the case of the restricted two body problem is the topic of the present talk

2-Body Problem (1PN)

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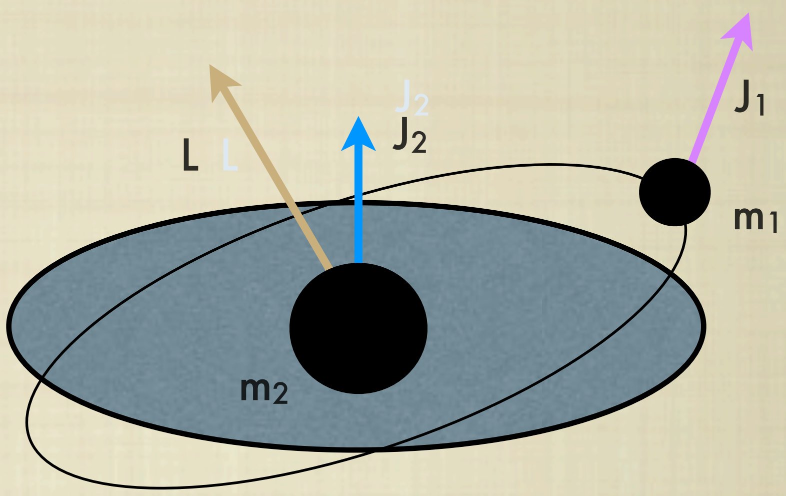

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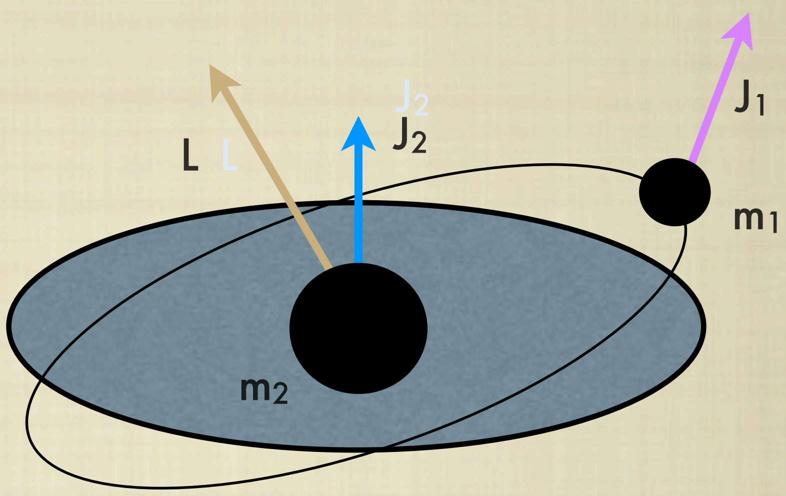
the generating Hamiltonian in the barycenter coordinate system is



$$\mathcal{H} = \frac{\mathbf{J}_1^2}{2I_1} + \frac{1}{2}\frac{\mathbf{p_1}^2}{m_1} - \frac{\mathcal{G}m_1m_2}{r} - \frac{1}{8c^2}\frac{\mathbf{p_1}^4}{m_1^3} - \frac{3\mathbf{p_1}^2}{2m_1c^2}\frac{\mathcal{G}m_2}{r}$$
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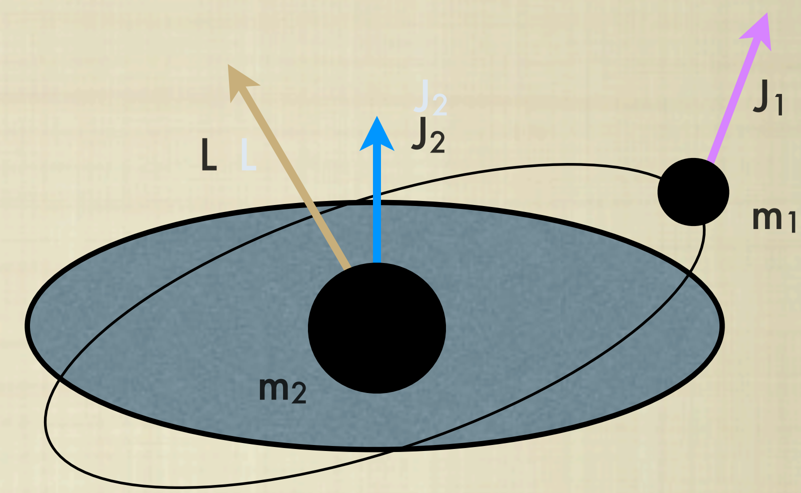

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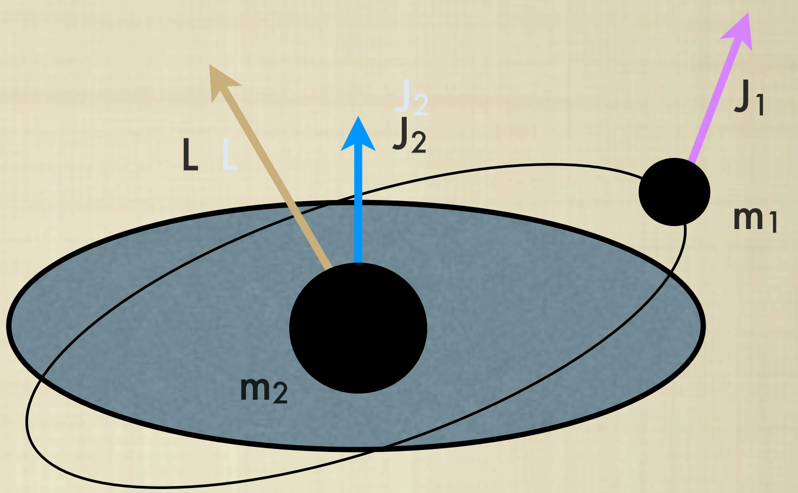

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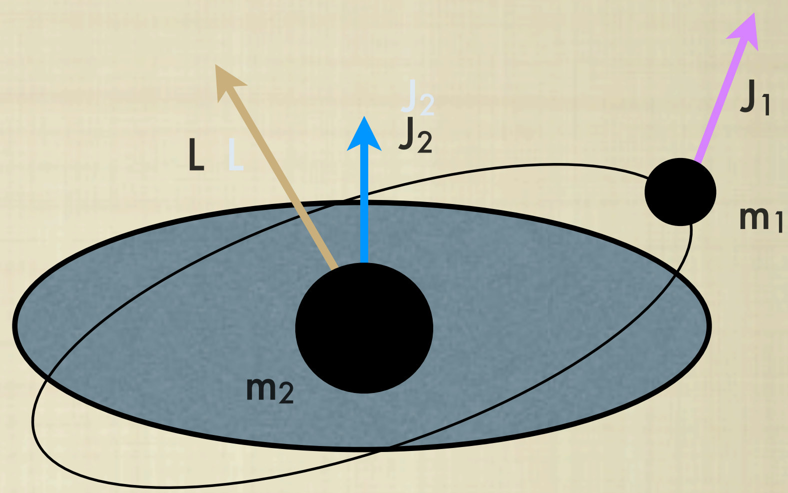

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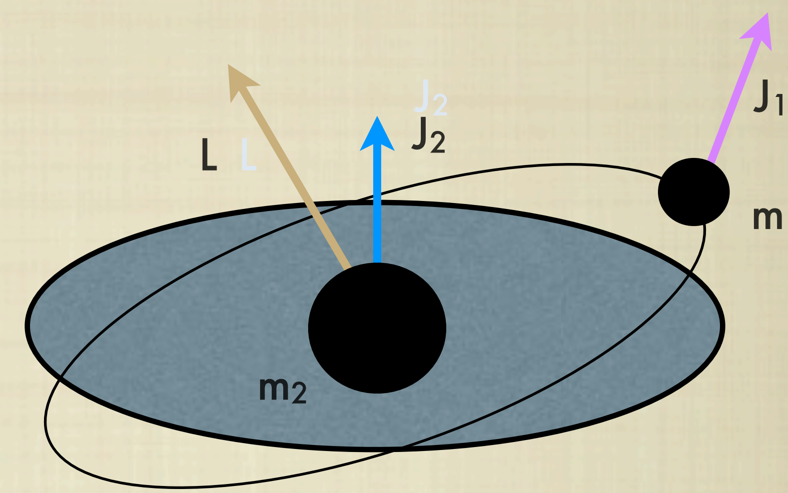

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\end{aligned}
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The two Spins "J" are defined as 3D Euclidean spin vectors (Damour et al. 2008).

We will assume J_2 constant in modulus and direction.

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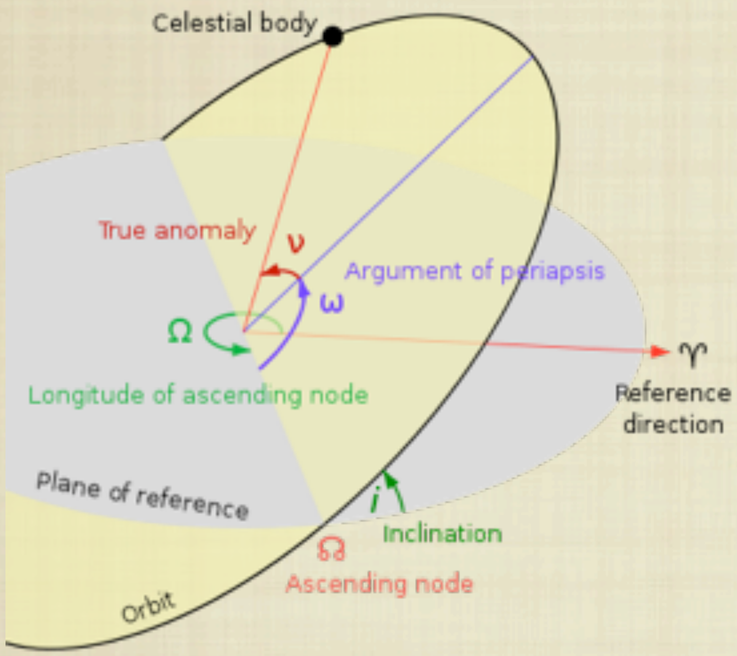
- Delaunay Variables



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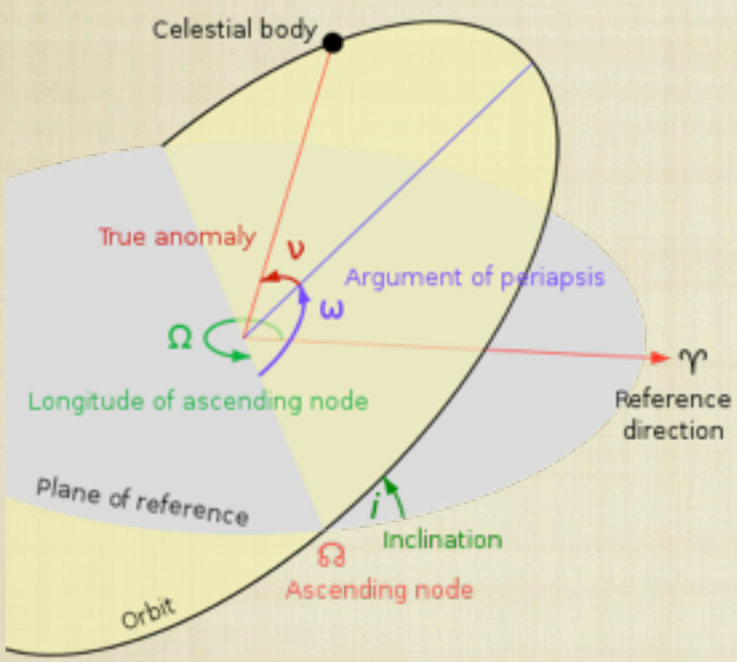
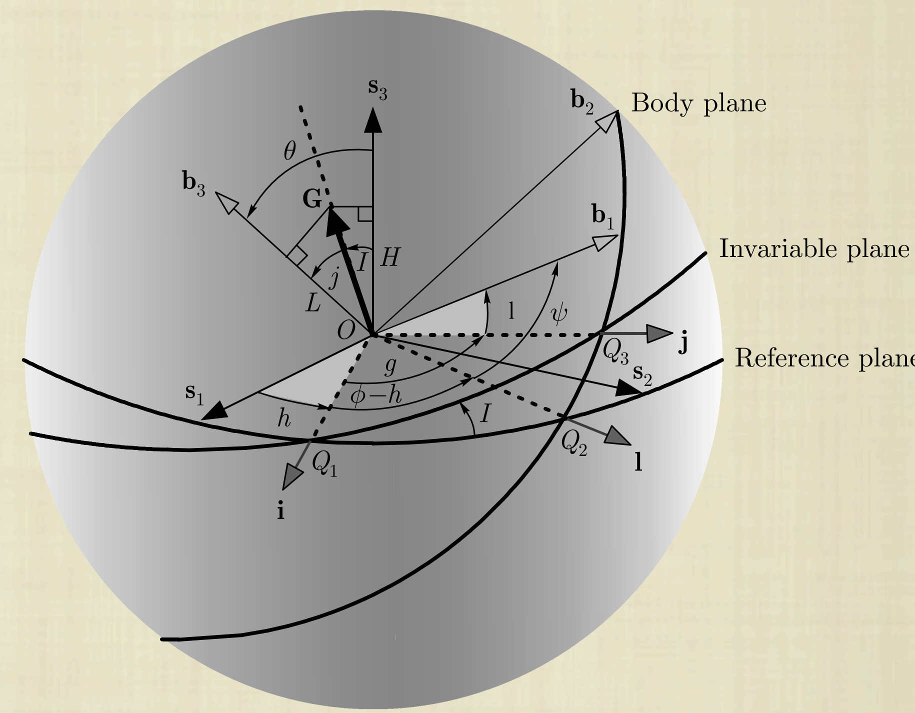
- Delaunay Variables
- Serret-Andoyer Variables





$$L = \sqrt{\mathcal{G} m a}, \quad G = L\sqrt{1-e^2}, \quad H = G\cos i,$$
$$l = M, \quad g = \omega, \quad h = \Omega.$$

$$\tilde{G} = \frac{J_1}{m_1}, \quad \tilde{H} = \frac{J_{1,z}}{m_1},$$
$$\tilde{h} = \arctan\left(\frac{\sqrt{J_{1,y}^2 + J_{1,z}^2}}{J_{1,x}}\right)$$

Action-Angle Variables

The Hamiltonian can then be written schematically as

$$\mathcal{H} = \mathcal{H}_0 + \frac{1}{c^2}\mathcal{H}_1,$$

where

$$\mathcal{H}_0 = \frac{1}{2}\mathcal{I}_1\tilde{G}^2 - \frac{\mathcal{G}^2 m_2^2}{2L^2},$$

$$\begin{aligned}\mathcal{H}_1 = {} & \mathcal{A}_0 + \frac{\mathcal{A}_1}{r} + \frac{\mathcal{A}_2}{r^2} + \frac{1}{r^3}\left(\mathcal{A}_{3a} + \mathcal{A}_{3b}\cos\left(\tilde{h} - h\right)\right) \\ & + \frac{1}{r^3}\left(\mathcal{B}_0 \cos\left(2f + 2g\right) + \mathcal{B}_1 \cos\left(2f + 2g + \tilde{h} - h\right)\right. \\ & \left. + \mathcal{B}_2 \cos\left(2f + 2g - \tilde{h} + h\right)\right),\end{aligned}$$

$$\mathcal{A}_i = \mathcal{A}_i(L, G, H, \tilde{G}, \tilde{H}) \qquad f = f(L, G, l)$$

$$\mathcal{B}_i = \mathcal{B}_i(L, G, H, \tilde{G}, \tilde{H}) \qquad r = r(L, G, l)$$

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With this method the Hamiltonian of the problem is simplified via a canonical transformation. The general Lie series transformation reads

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- The above equation represent an average over the mean motion
- its resolution simplifies the Hamiltonian in the sense given above
- its results is valid even for highly eccentric orbits (but still elliptic)

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With the choice above and making a (canonical) change of variables

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$$\mathcal{F}_0 = \frac{3\mathcal{G}^4 m_2^4}{8G^3L^4}\left[4HL\tilde{H}_* + 5G^3 - 4L\left(2G^2 + H^2\right)\right]$$
$$+ \frac{J_2\mathcal{G}^4 m_2^3}{2G^5L^3}\left[\tilde{H}_*\left(G^2 - 3H^2\right) + 3H\left(G^2 + H^2\right)\right]$$
$$\mathcal{F}_1 = \frac{3\mathcal{G}^4 m_2^3 G_{xy}\tilde{G}_{xy*}}{2G^5L^3}\left(G^2 m_2 + HJ_2\right).$$

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This Hamiltonian depends only on one angle and **it is integrable.**

The Hamilton Eqs.

The hamilton equations have the following form

$$\frac{dL}{dt} = 0, \qquad \frac{dl}{dt} = \frac{\mathcal{G}^2 m_2^2}{L^3} + \epsilon\left(\frac{\partial \mathcal{F}_0}{\partial L} + \frac{\partial \mathcal{F}_1}{\partial L}\cos h_*\right),$$

$$\frac{dG}{dt} = 0, \qquad \frac{dg}{dt} = \epsilon\left(\frac{\partial \mathcal{F}_0}{\partial G} + \frac{\partial \mathcal{F}_1}{\partial G}\cos h_*\right),$$

$$\frac{dH}{dt} = \epsilon \mathcal{F}_1 \sin h_*, \qquad \frac{dh_*}{dt} = \epsilon\left(\frac{\partial \mathcal{F}_0}{\partial H} + \frac{\partial \mathcal{F}_1}{\partial H}\cos h_*\right),$$

$$\frac{d\tilde{G}}{dt} = 0, \qquad \frac{d\tilde{g}}{dt} = \mathcal{I}_1 \tilde{G} + \epsilon \frac{\partial \mathcal{F}_1}{\partial \tilde{G}}\cos h_*,$$

$$\frac{d\tilde{H}_*}{dt} = 0, \qquad \frac{d\tilde{h}}{dt} = \epsilon\left(\frac{\partial \mathcal{F}_0}{\partial \tilde{H}_*} + \frac{\partial \mathcal{F}_1}{\partial \tilde{H}_*}\cos h_*\right).$$

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\end{aligned}
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\frac{dl}{dt} &= \frac{\mathcal{G}^2 m_2^2}{L^3} + \epsilon \left(\frac{\partial \mathcal{F}_0}{\partial L} + \frac{\partial \mathcal{F}_1}{\partial L} \cos h_* \right), \\
\frac{dg}{dt} &= \epsilon \left(\frac{\partial \mathcal{F}_0}{\partial G} + \frac{\partial \mathcal{F}_1}{\partial G} \cos h_* \right), \\
\frac{dh_*}{dt} &= \epsilon \left(\frac{\partial \mathcal{F}_0}{\partial H} + \frac{\partial \mathcal{F}_1}{\partial H} \cos h_* \right), \\
\frac{d\tilde{g}}{dt} &= \mathcal{I}_1 \tilde{G} + \epsilon \frac{\partial \mathcal{F}_1}{\partial \tilde{G}} \cos h_*, \\
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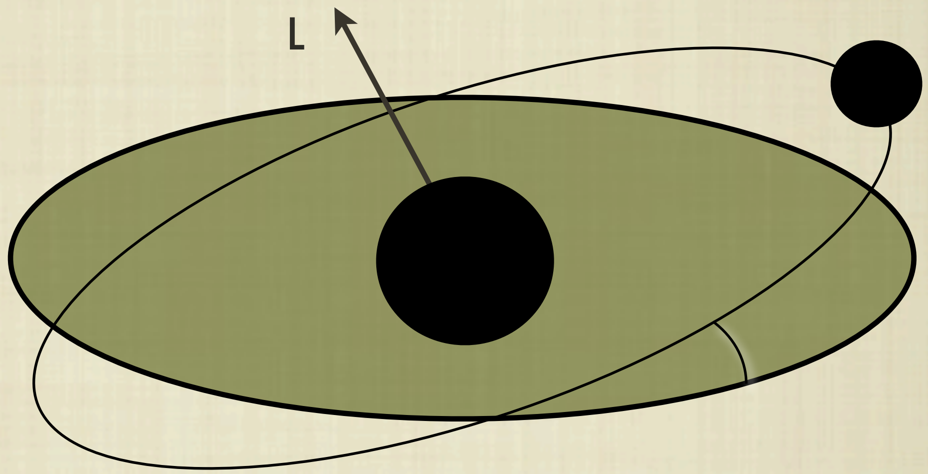
All the momenta apart H are conserved.

The conservation of $\tilde{H}_*$ implies the conservation of the z component of the total angular momentum.

Einstein Precession

In the case of absence of spin all momenta are constants of motion and the equations for the averaged orbital coordinates are:

$$\frac{dl}{dt} = \frac{\mathcal{G}^2 m_2^2}{L^3} + \epsilon \frac{3\mathcal{G}^4 m_2^4}{2GL^5}(6L - 5G),$$
$$\frac{dg}{dt} = 3\epsilon \frac{m_2^4 \mathcal{G}^4}{L^3 G^2},$$
$$\frac{dh}{dt} = 0.$$



The second equation gives the classical formula

$$\frac{dg}{dt} \equiv \frac{d\omega}{dt} = \frac{3 m_2^{\frac{3}{2}} \mathcal{G}^{\frac{3}{2}}}{c^2 a^{\frac{5}{2}} \left(1 - e^2\right)},$$

Lens-Thirring effect

If only the central body is rotating, all momenta are constants of motion and

$$\frac{dl}{dt} = \frac{\mathcal{G}^2 m_2^2}{L^3} + \epsilon\left(\frac{3\mathcal{G}^4 m_2^4}{2GL^5}(6L-5G) - \frac{6\mathcal{G}^4 H J_2 m_2^3}{G^3 L^4}\right),$$
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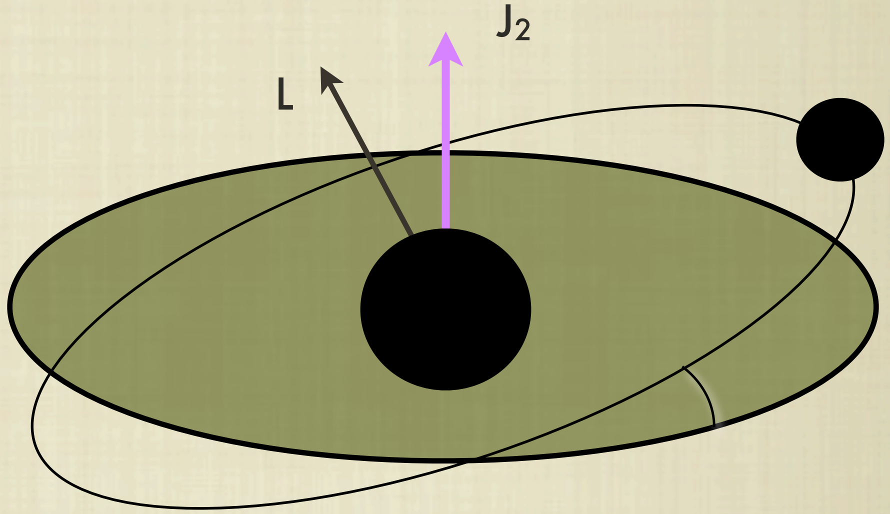

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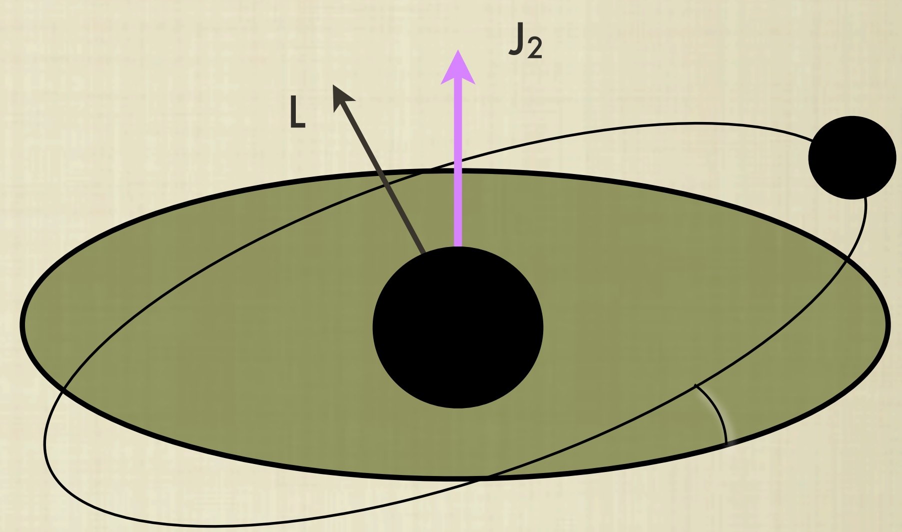

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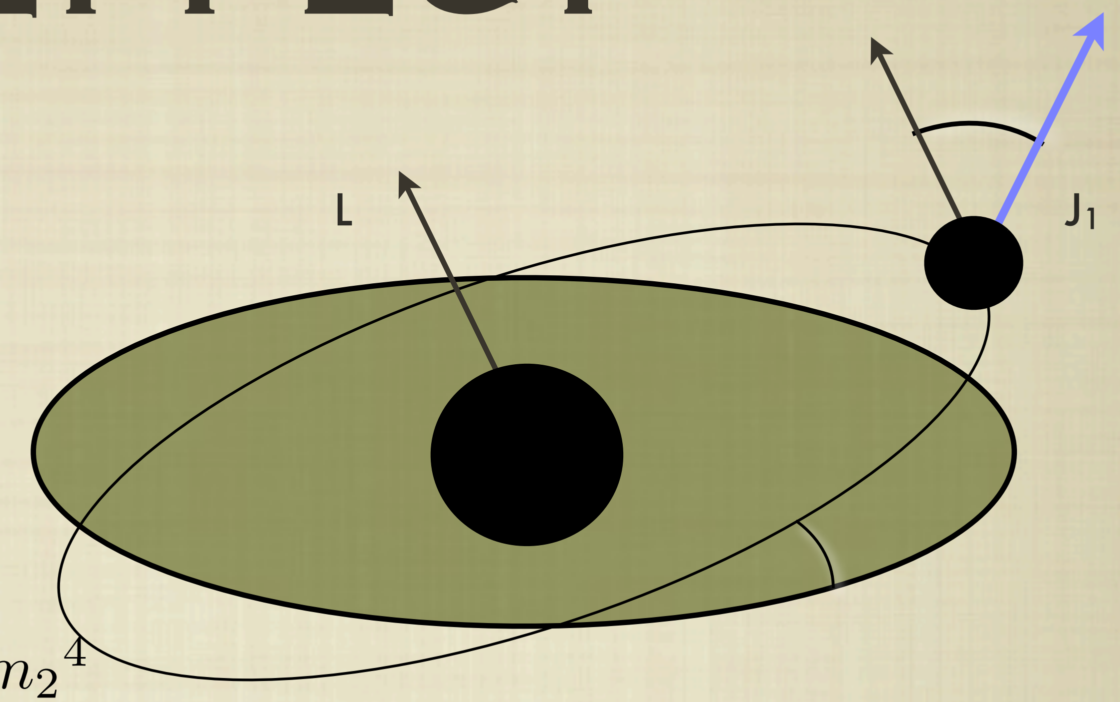

If only the secondary body is rotating, we have

$$\frac{dH}{dt} = \frac{3}{2}\epsilon\frac{G_{xy}\mathcal{G}^4\tilde{G}_{xy*}{m_2}^4}{G^3L^3}\sin h_*,$$

$$\frac{dh_*}{dt} = \epsilon\left[-3\frac{H\mathcal{G}^4{m_2}^4}{G^3L^3} + \frac{3}{2}\frac{\mathcal{G}^4\tilde{H}_*{m_2}^4}{G^3L^3} + \left(-\frac{3}{2}\frac{G_{xy}H\mathcal{G}^4{m_2}^4}{G^3L^3\tilde{G}_{xy*}} + \frac{3}{2}\frac{G_{xy}\mathcal{G}^4\tilde{H}_*{m_2}^4}{G^3L^3\tilde{G}_{xy*}} - \frac{3}{2}\frac{H\mathcal{G}^4\tilde{G}_{xy*}{m_2}^4}{G^3G_{xy}L^3}\right)\cos h_*\right].$$

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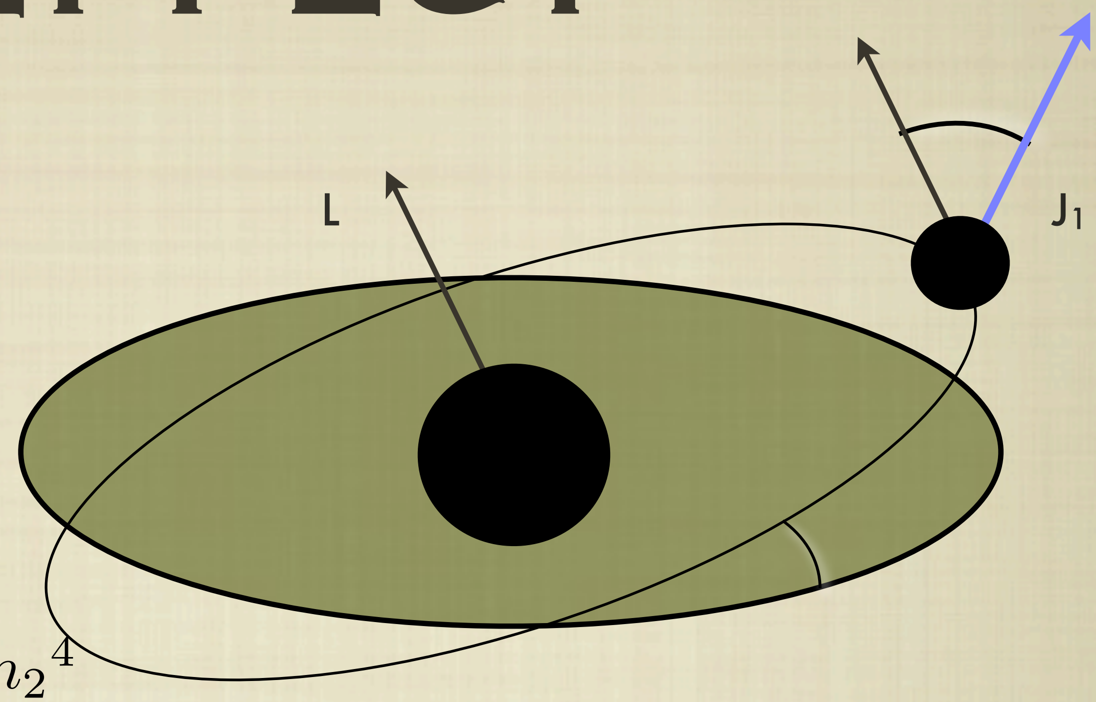

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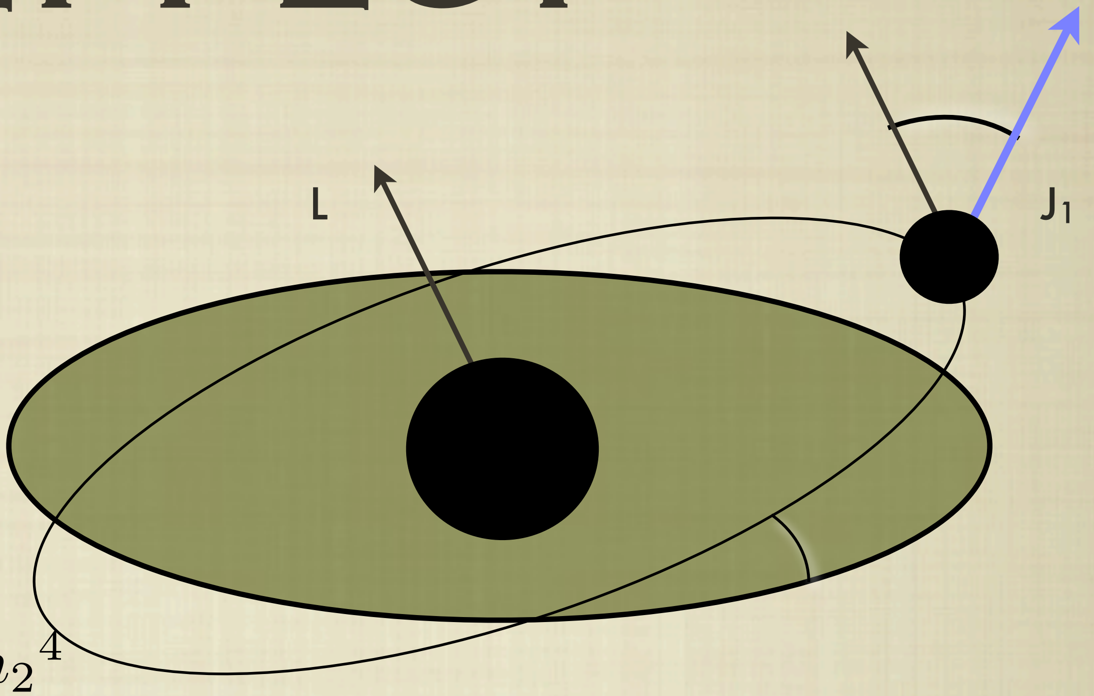

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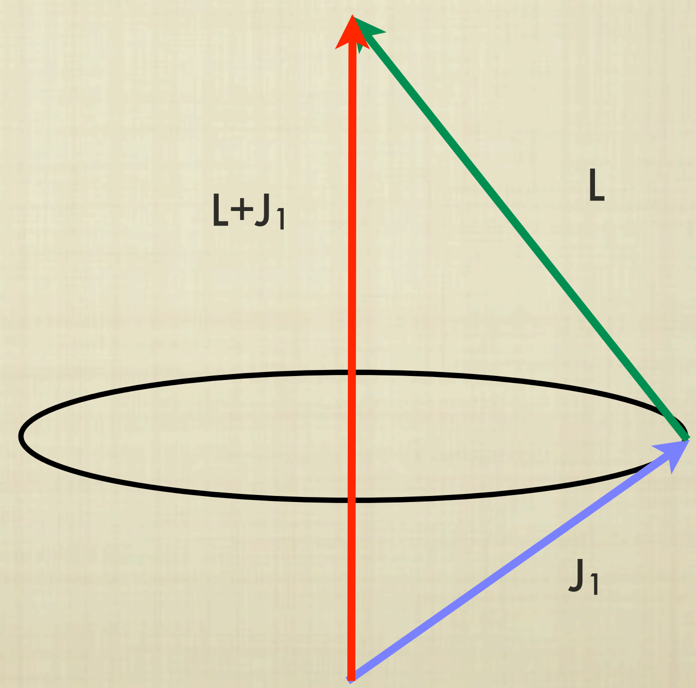

However the conservation of the other momenta indicate that:

Geodetic Effect

We can analyze the phase space of the system above.

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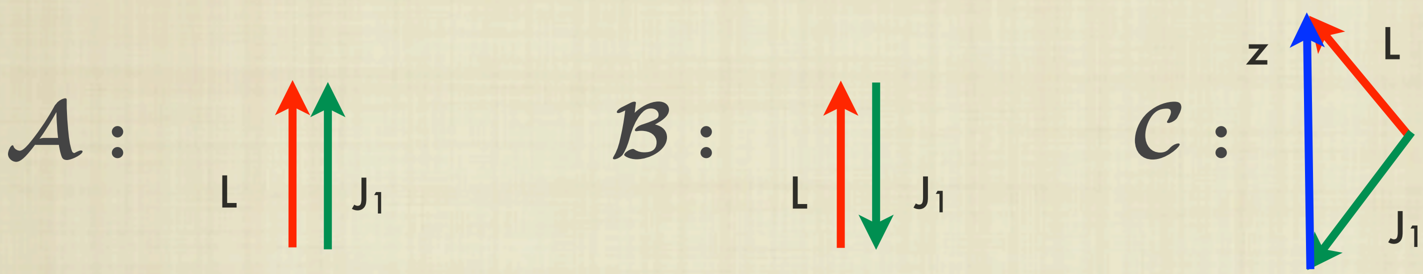
We find in general three fixed points:



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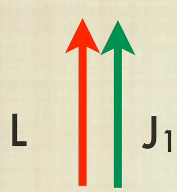

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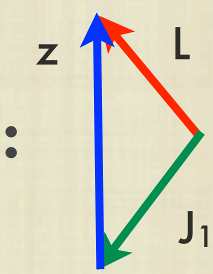
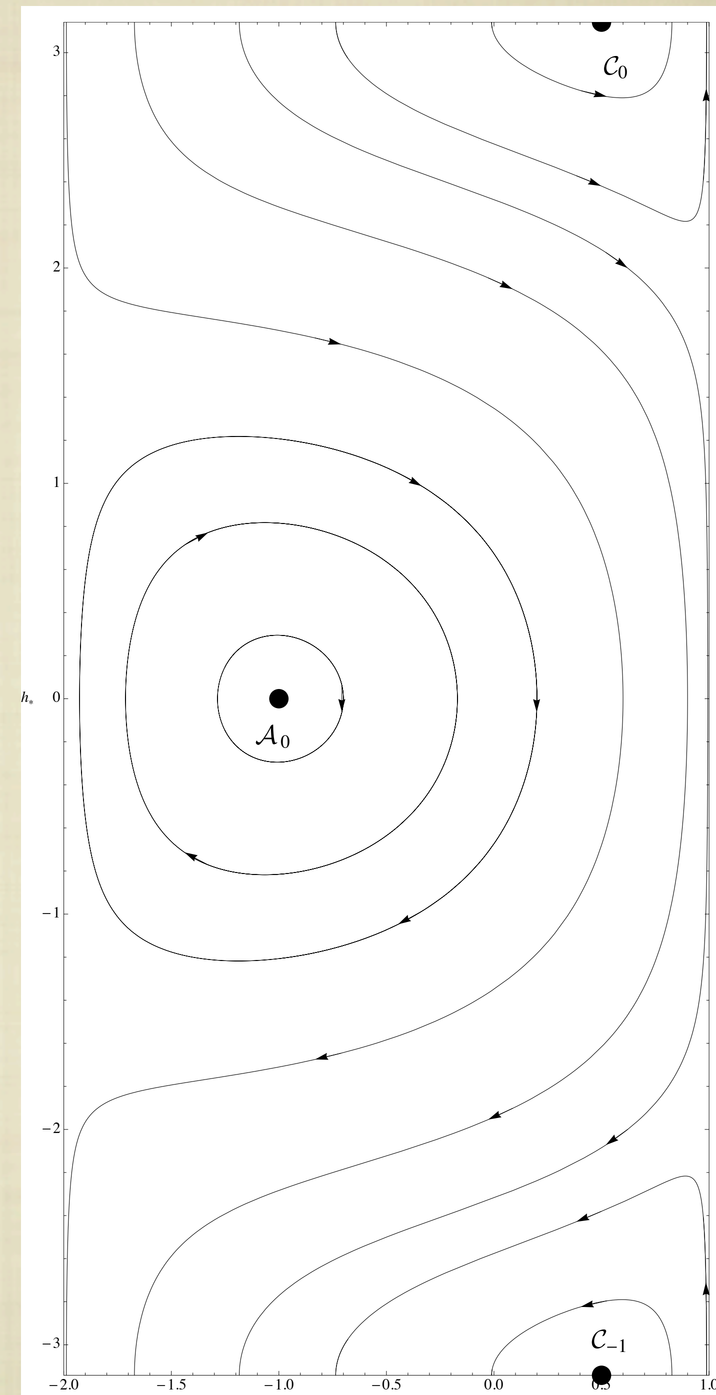


...but for a given set of parameters only two fixed points appear in the phase space.



Geodetic Effect

In the case in which the spin is much smaller than the angular momentum

$$\frac{d\tilde{h}}{dt} = \epsilon \frac{3m_2^4 \mathcal{G}^4}{2L^3G^2} = \frac{3\left(m_2\mathcal{G}\right)^{\frac{3}{2}}}{2c^2a^{\frac{5}{2}}\left(1-e^2\right)},$$

which is the classical formula of the geodetic effect.

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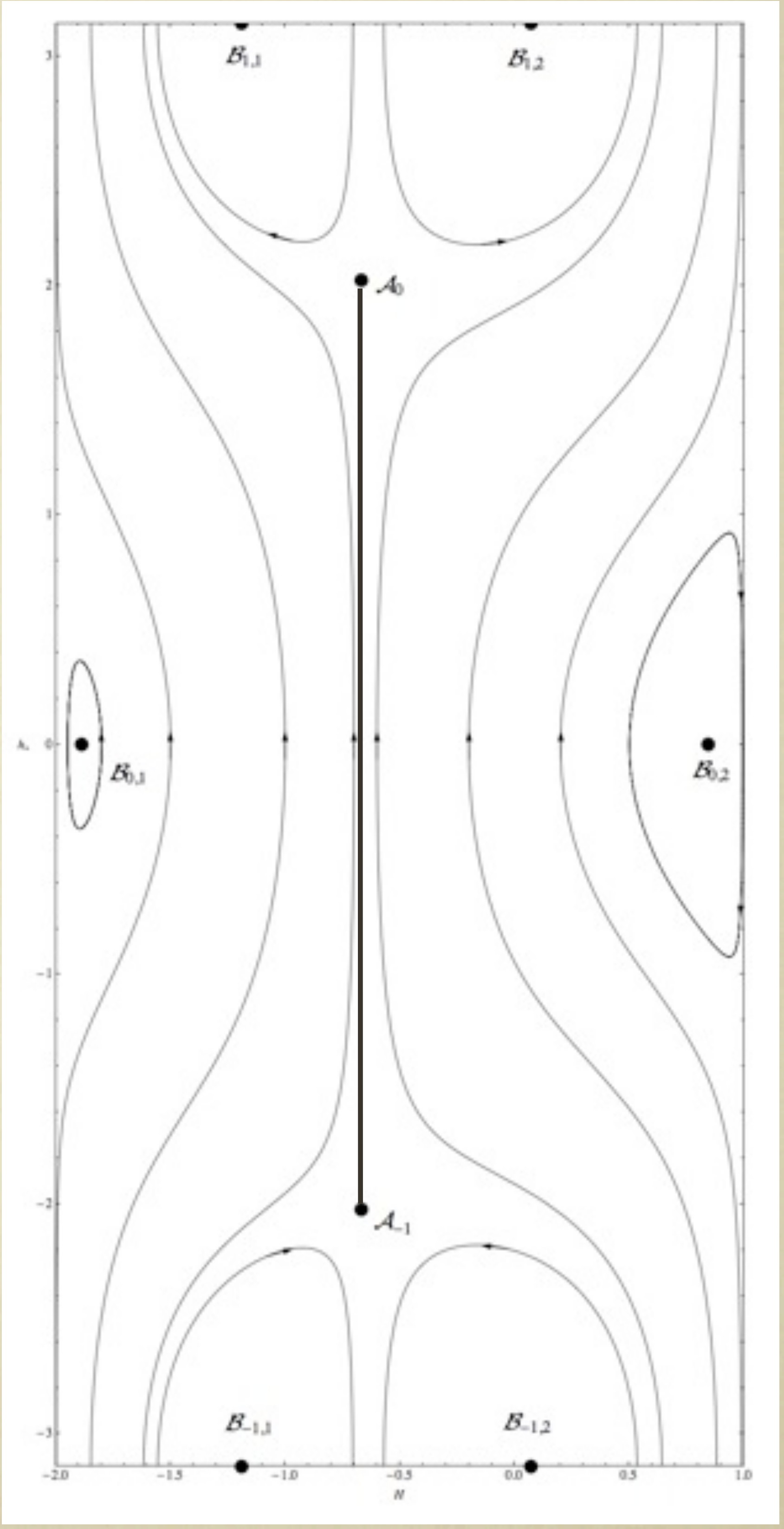

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- $$H^{(e)} = \frac{G^2}{\mathcal{J}_2}, \quad \cos h_*^{(e)} = \frac{G^2\left(\frac{G^2}{\mathcal{J}_2} + \mathcal{J}_2 - \tilde{H}_*\right)}{\mathcal{J}_2 \tilde{G}_{xy*} G_{xy}}.$$
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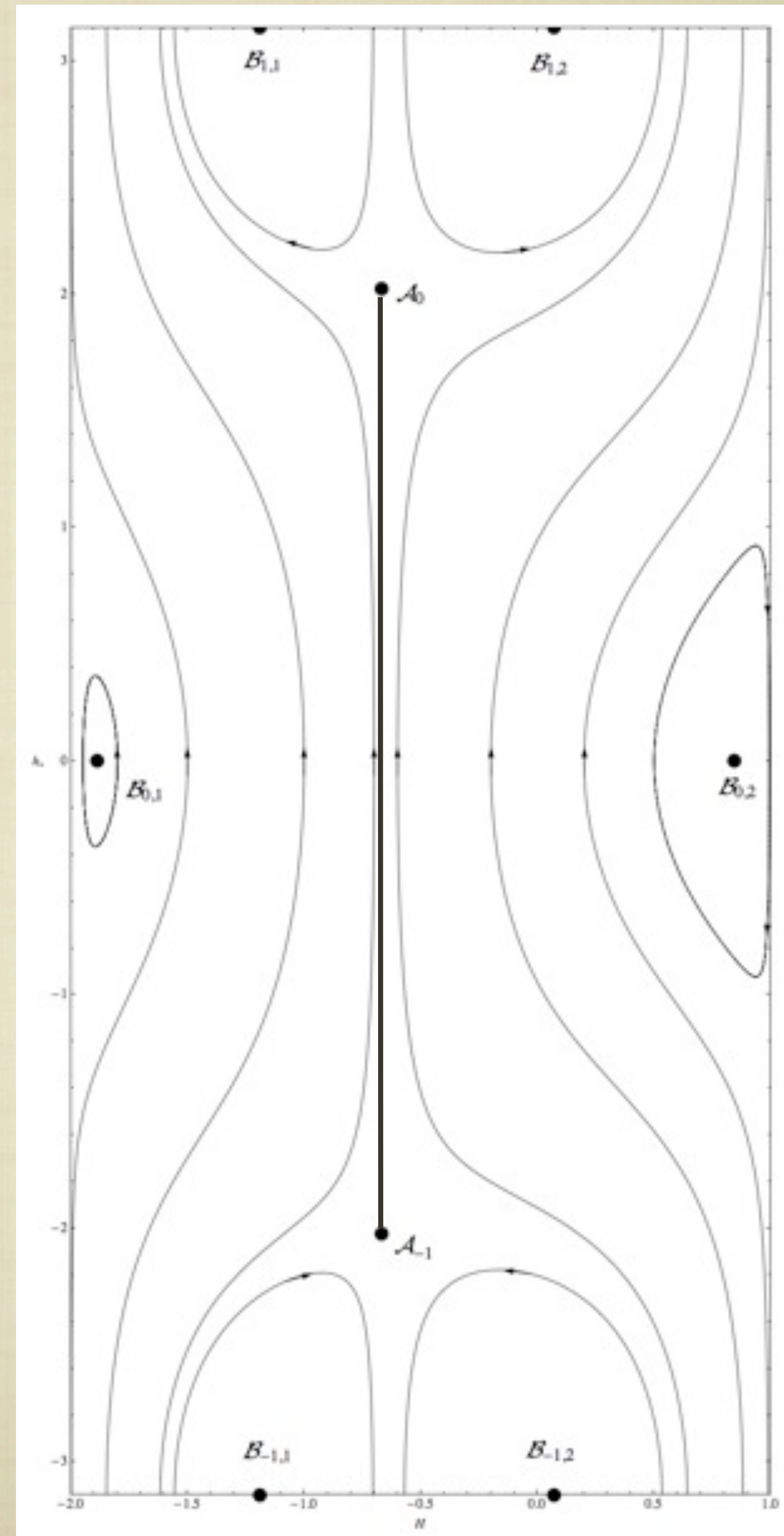

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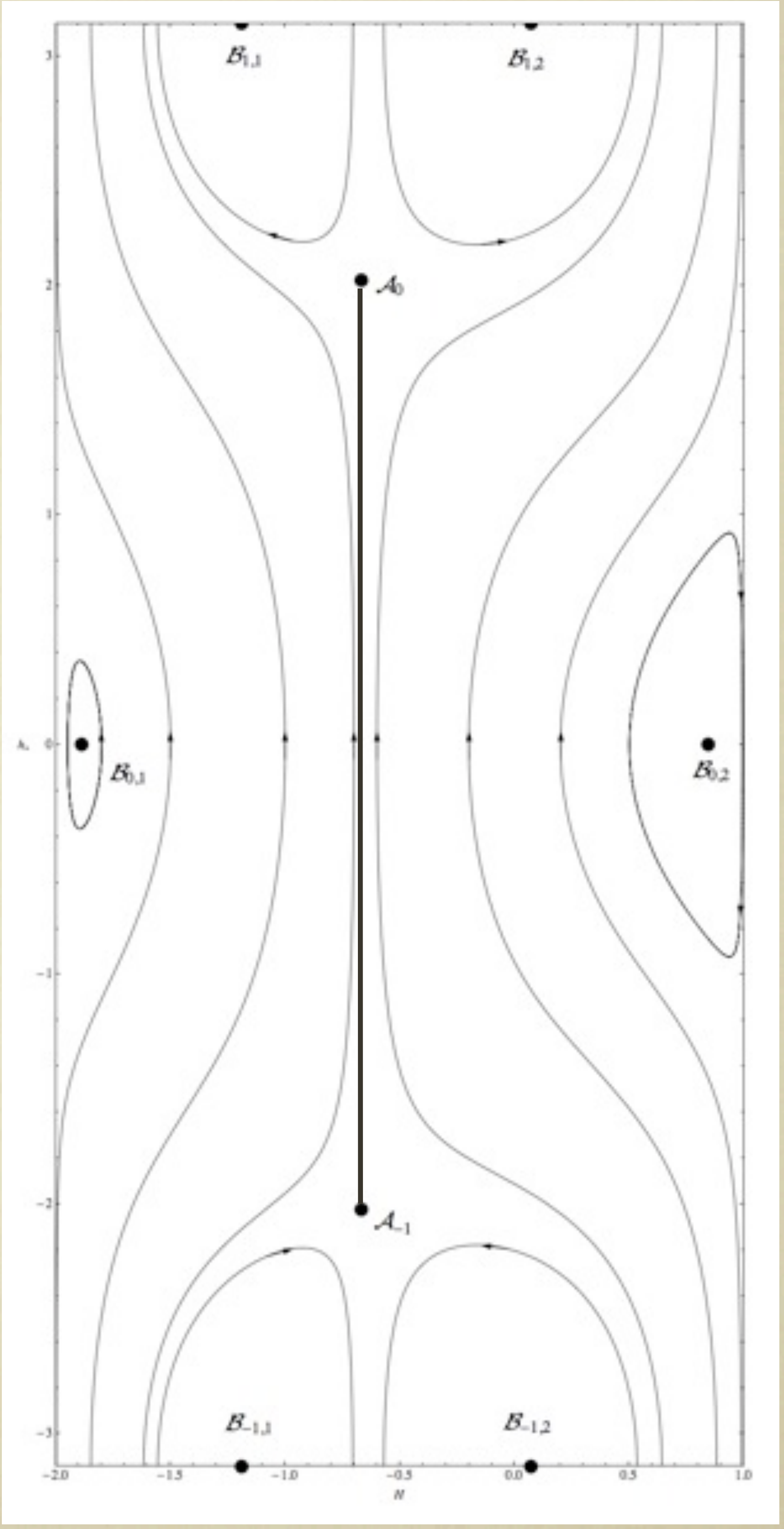

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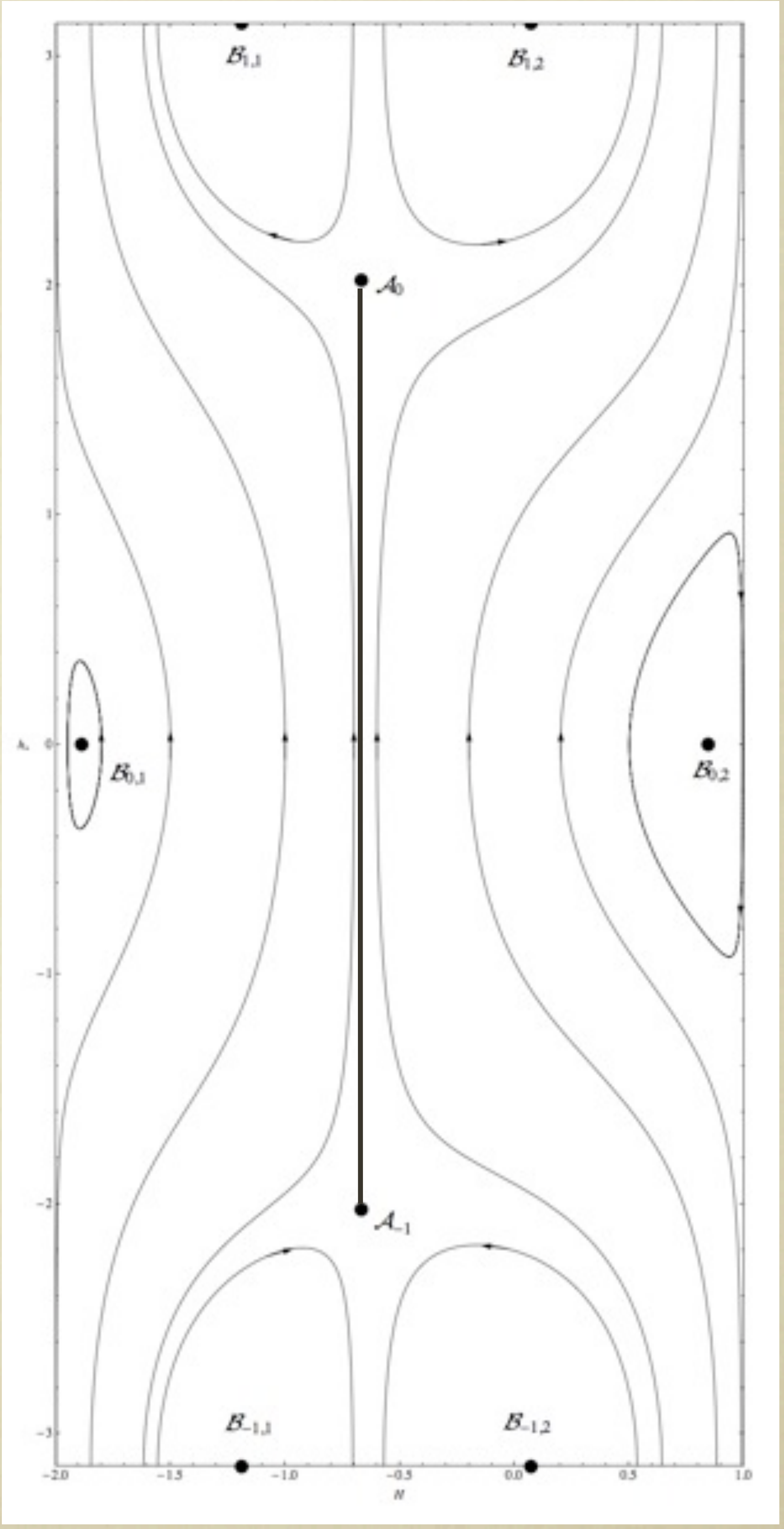

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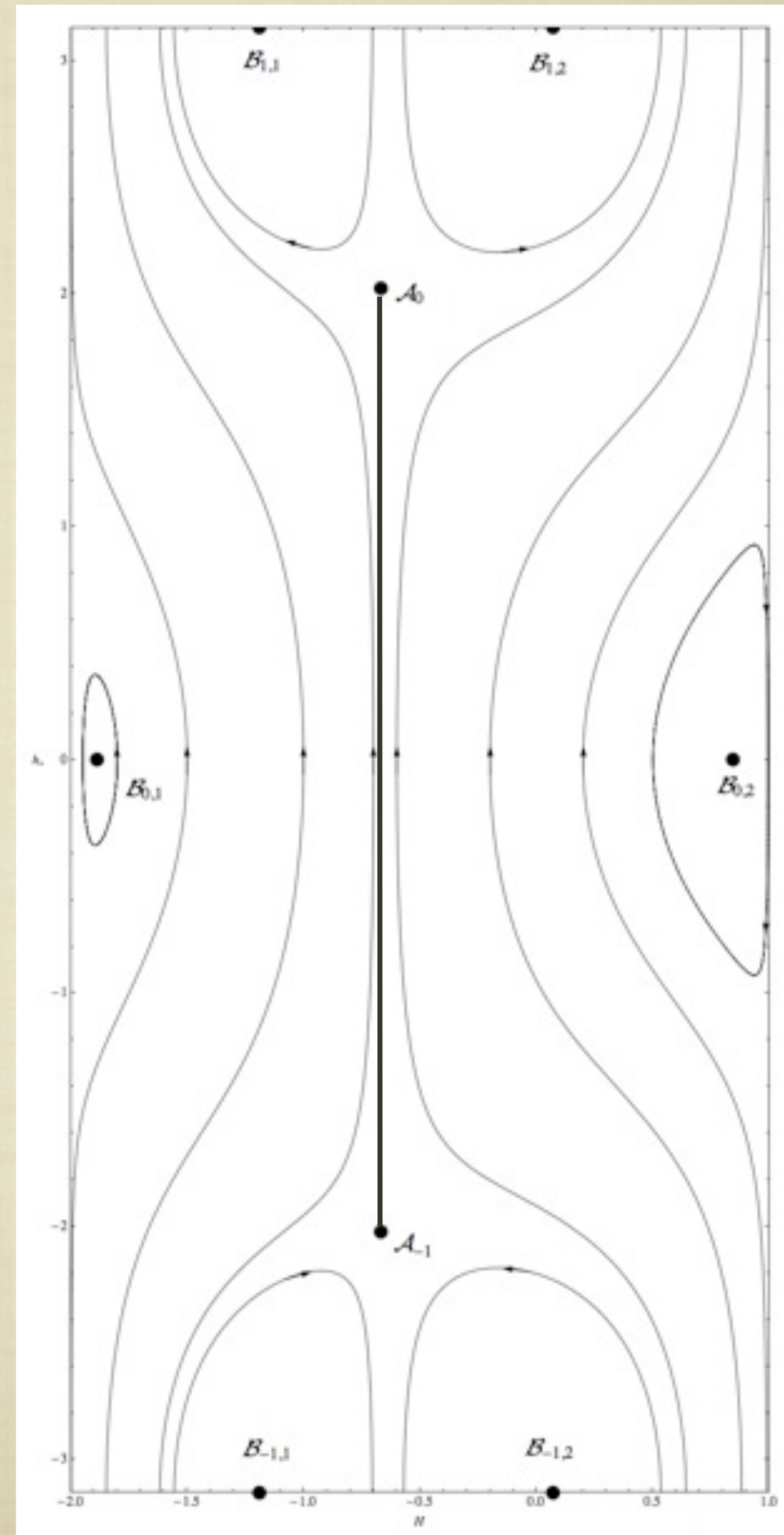

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Combining the HAmiltonian with the equation for H, one has

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$$H(t) = H_0 + \frac{\frac{1}{2}f_4'(H_0)\left[\wp(t) - \frac{1}{24}f_4''(H_0)\right] + \frac{1}{24}f_4(H_0)\,f_4'''(H_0) \pm \sqrt{f_4(H_0)}\wp'(t)}{2\left[\wp(t) - \frac{1}{24}f_4''(H_0)\right]^2 - \frac{1}{48}f_4(H_0)\,f_4^{iv}(H_0)},$$

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where $\wp(t)$ is the Weierstrass elliptic function.

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- In the case $J_2 = 0$ the general solution reduces to a periodic function with angular velocity

$$\Omega = \frac{3}{2}\epsilon\frac{m_2^4\mathcal{G}^4}{L^3G^3}M$$

Numerical Examples

In the case of a Mercury-like planet we have

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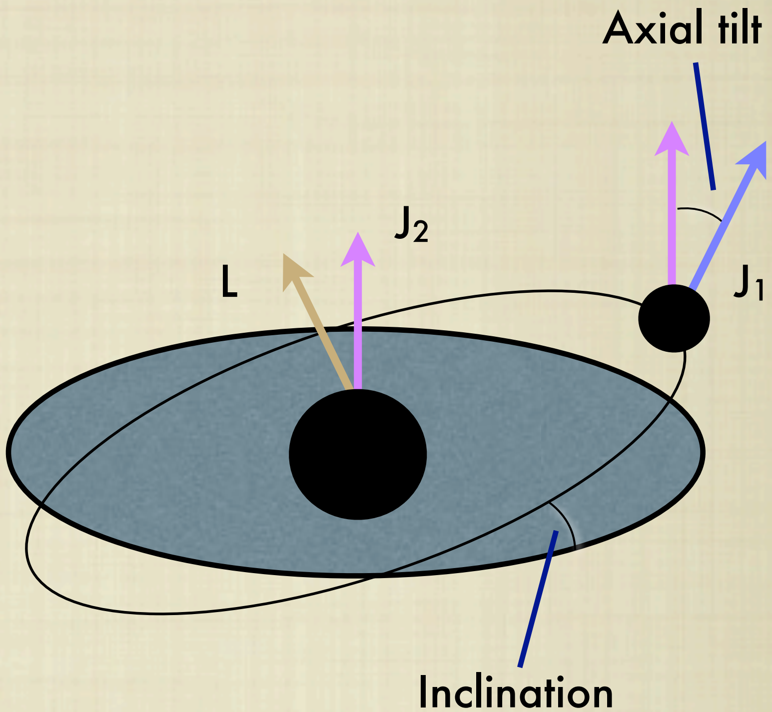

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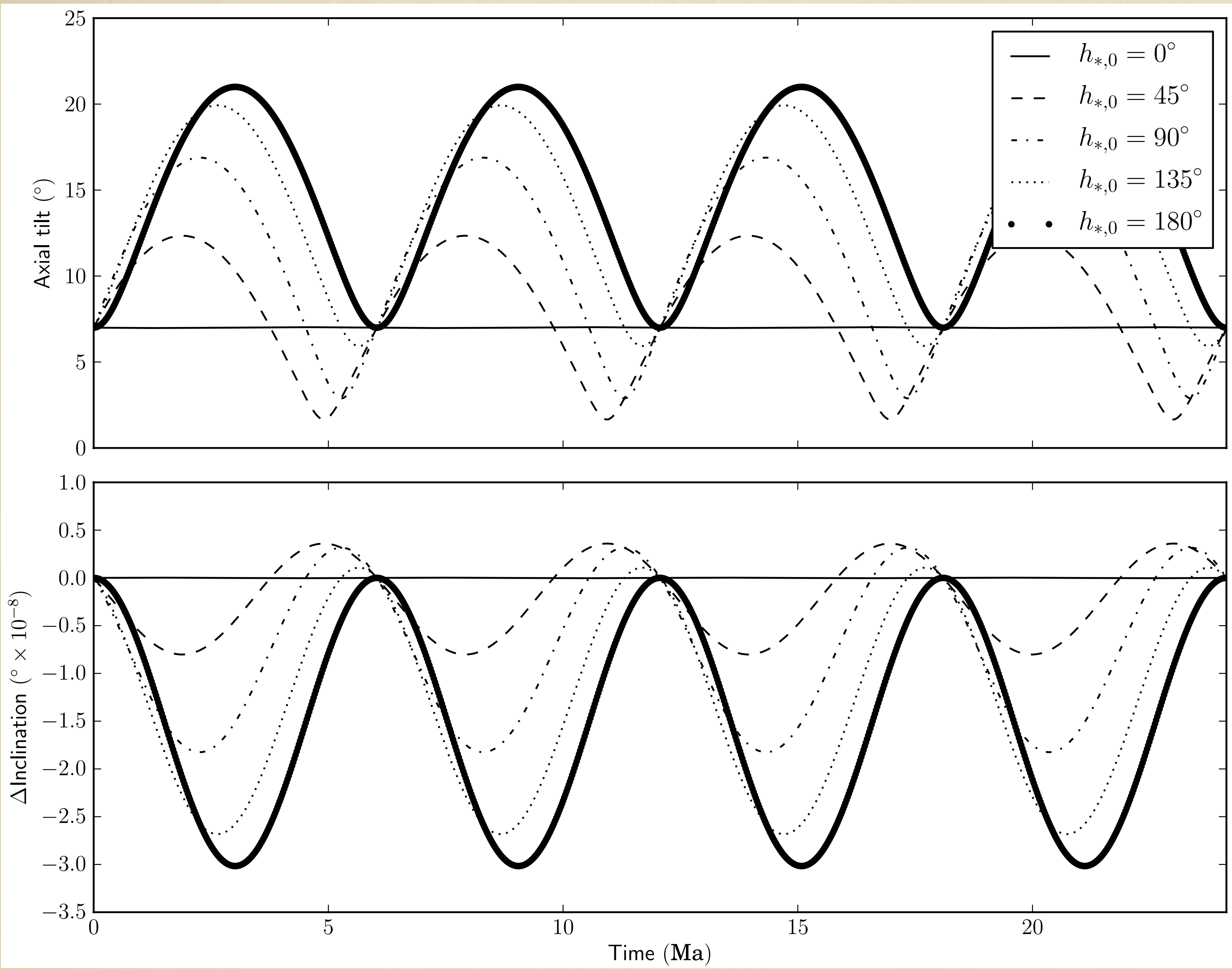

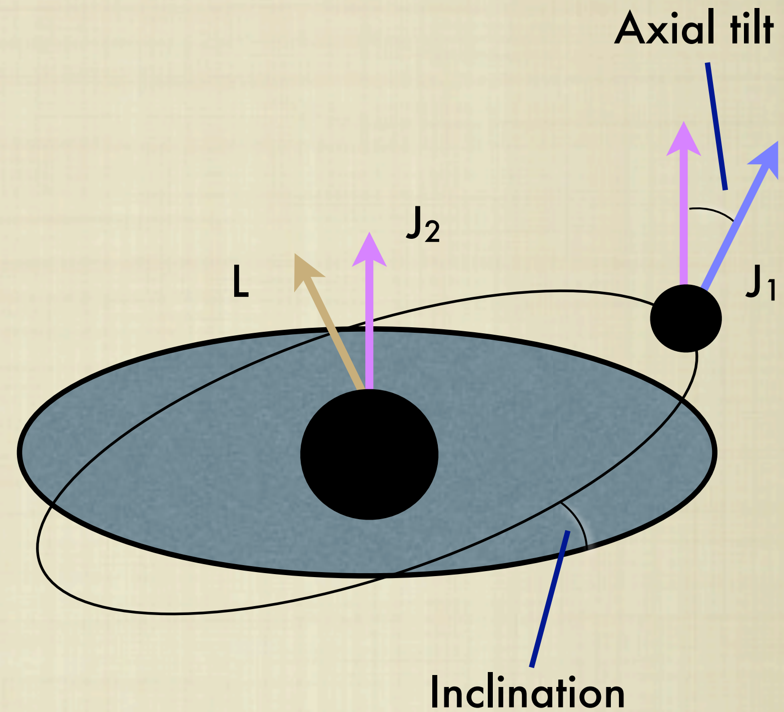

Parameter	Value (SI units)
L_0	2.77×10^{15}
G_0	2.71×10^{15}
H_0	2.69×10^{15}
$\tilde{G}_0$	2.95×10^{6}
$\tilde{H}_0$	2.93×10^{6}
J_2	1.12×10^{42}
r_1	6.37×10^{6}

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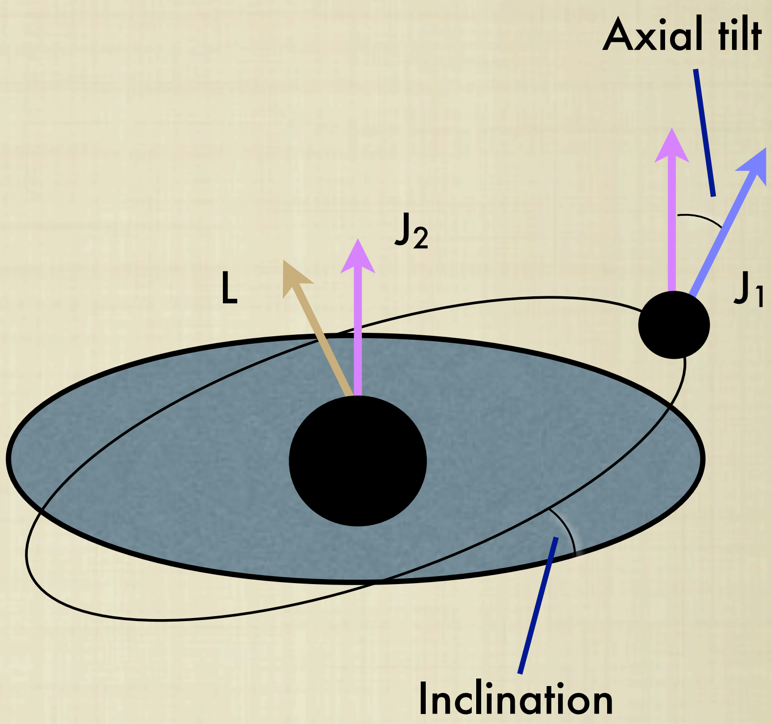

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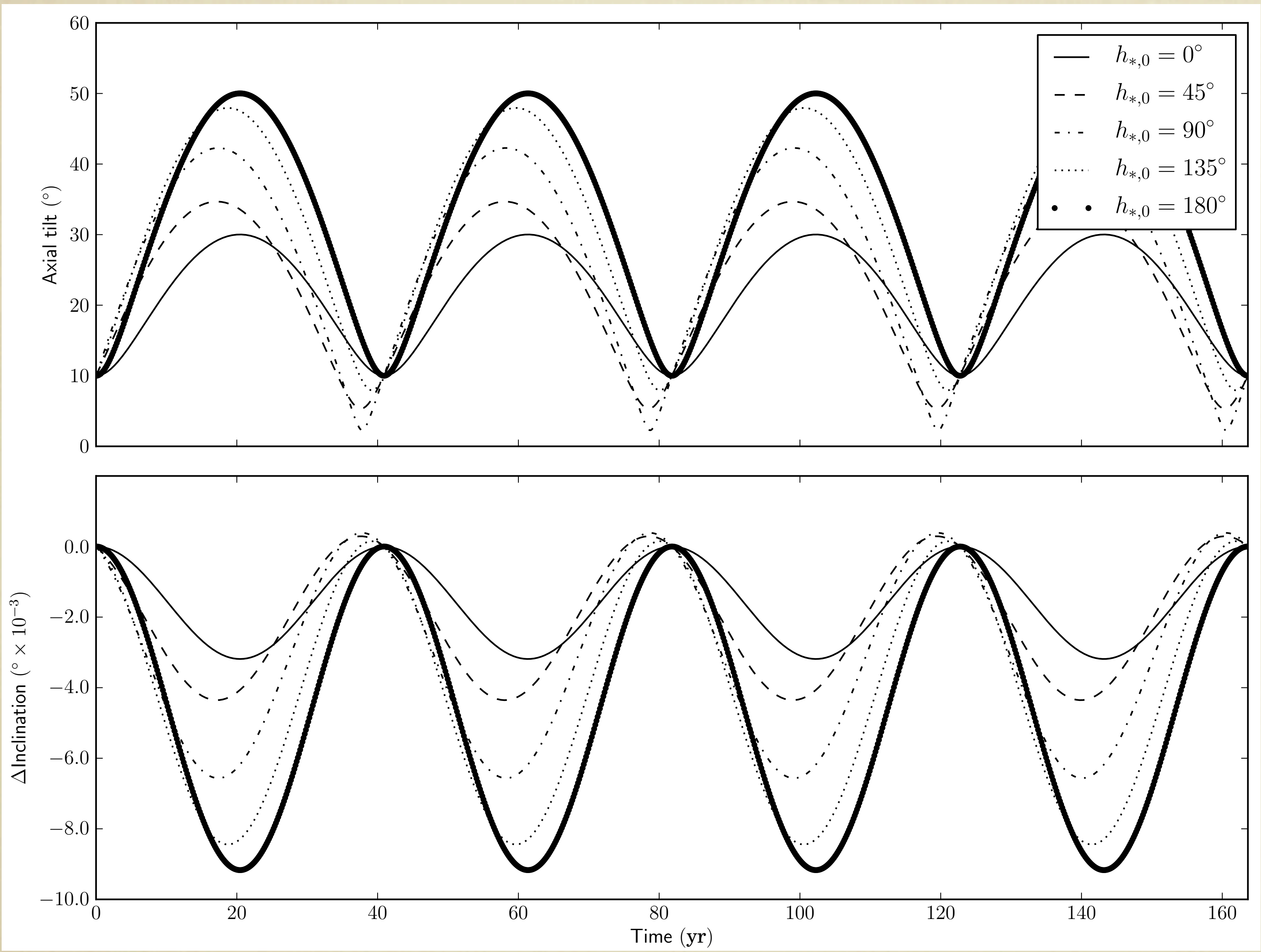

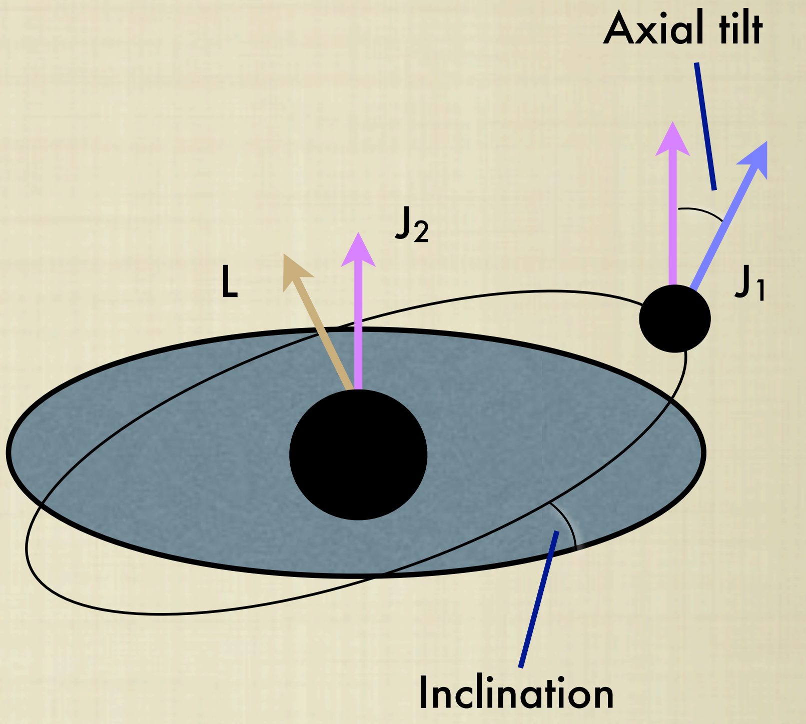

Parameter	Value (SI units)
L_0	3.33×10^{14}
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H_0	3.12×10^{14}
$\tilde{G}_0$	5.32×10^{10}
$\tilde{H}_0$	5.23×10^{10}
J_2	4.83×10^{41}
r_1	2.76×10^{7}

Conclusion

- We have analyzed the restricted 2-body problem at the 1PN approximation using Lie perturbation theory;
- Our approach allows a complete (classical) analysis of the problem and the deduction of the exact solution of the problem of motion;
- Our results match and generalize all the ones already found for the Einstein precession, the Lens-Thirring effect and the geodetic effect;
- For particular values of the parameters the solution for the system can acquire a non- periodic character.

The RPS connection

The possibility to solve exactly the restricted 2-body problem has an impact in terms of RPS:

- In its present form our results can lead a semiclassical approach to the problem of RPS (is it useful?)
- Fits well with the ABC way of constructing RPS and allows a first exploration of more realistic spacetimes (rotating fields, inhomogeneous fields, etc.)
- It opens the explicit possibility to test General Relativity using GPS satellites (introduce PPN parameters)


IN TEN FEET TURN LEFT THEN TURN RIGHT
YOU HAVE ARRIVED AT YOUR DESTIN-ATION!
WILL YOU SHUTUP ALREADY! IM JUST GOING TO THE BATHROOM!