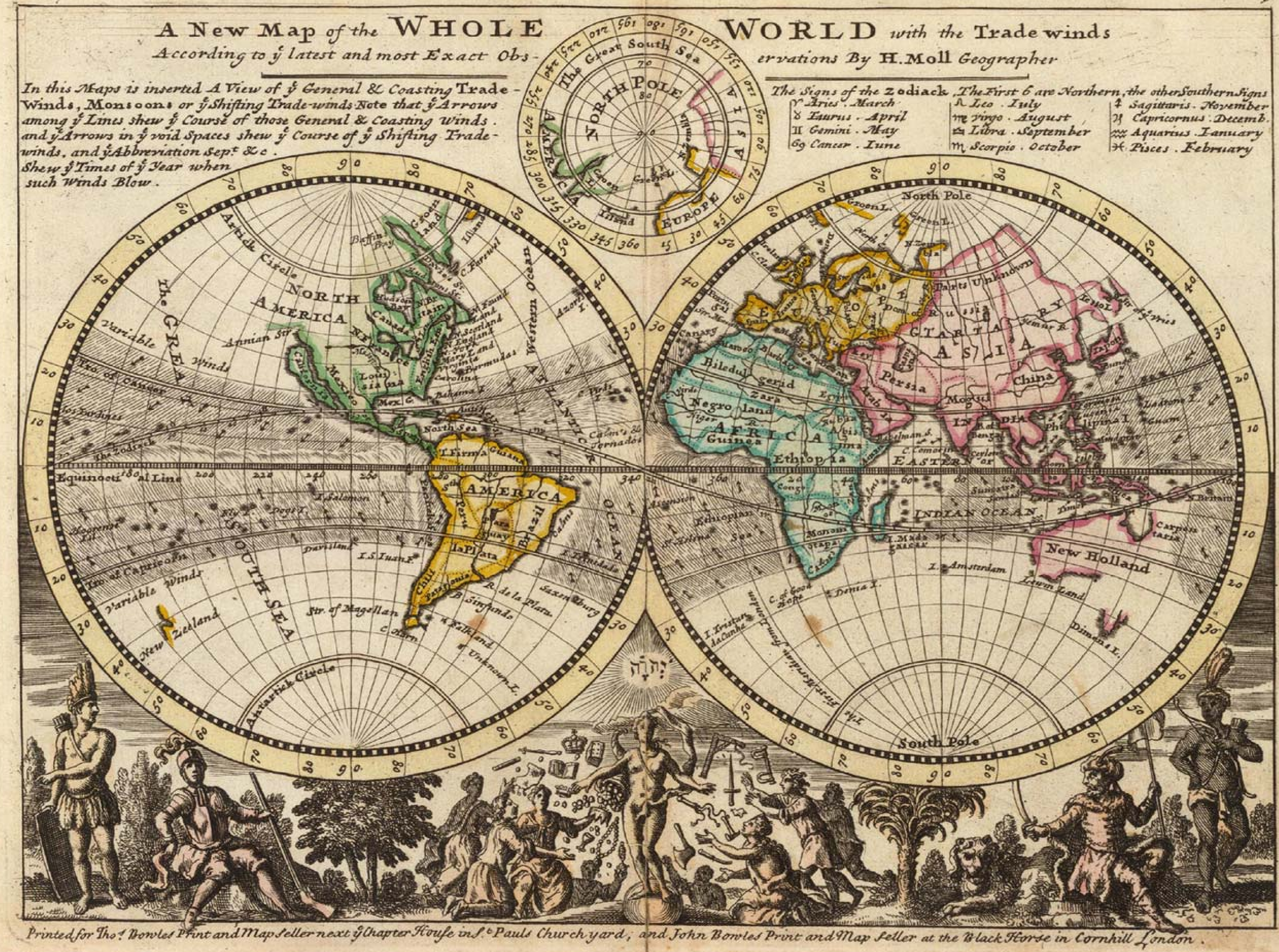
The concept of autonomous basis of coordinates (ABC) and its use in determining positions in space-time

Andrej Čadež

The use of coordinates through time

- To find a place on Earth
- To find a star or a planet in the sky
- To understand the laws of the Universe


A New Map of the WHOLE WORLD with the Trade winds
According to ye latest and most Exact Observations By H. Moll Geographer
In this Maps is inserted A View of ye General & Coasting Trade-Winds, Monsoons or ye Shifting Trade-winds Note that ye Arrows among ye Lines shew ye Course of those General & Coasting Winds. and ye Arrows in ye void Spaces shew ye Course of ye Shifting Trade-winds. and ye Abbreviation Sept. &c. Shew ye Times of ye Year when such Winds Blow.
The Signs of the Zodiack, The First 6 are Northern, the other Southern Signs
♈ Aries . March
♉ Taurus . April
♊ Gemini . May
♋ Cancer . June
♌ Leo . July
♍ Virgo . August
♎ Libra . September
♏ Scorpio . October
♐ Sagittaris . November
♑ Capricornus . Decemb.
♒ Aquarius . January
♓ Pisces . February
The Great South Sea
NORTH POLE
AMERICA
ASIA
EUROPE
NORTH AMERICA
SOUTH AMERICA
Western Ocean
ATLANTICK OCEAN
SOUTH SEA
Equinoctial Line
Tropick of Cancer
Tropick of Capricorn
Variable Winds
Arctick Circle
Antartick Circle
North Pole
South Pole
EUROPE
ASIA
AFRICA
Ethiopia
INDIAN OCEAN
New Holland
New Zeeland
China
Persia
India
Str. of Magellan
C. of Good Hope
Printed for Thos. Bowles Print and Map seller next ye Chapter House in St. Pauls Church yard, and John Bowles Print and Map seller at the Black Horse in Cornhill London.

axioms

- **Absolute space**, in its own nature, without relation to anything external, remains always similar and immovable. (Newton)
- **Absolute**, true, and mathematical **time**, of itself, and from its own nature, flows equably without relation to anything external. (Newton)
- More than other people he (Kepler) was a person of independent genius, sharp, and had in his hands the **motion of the earth**. (Galileo)

A frame in absolute space

- The **International Celestial Reference Frame** (ICRF) is a quasi-inertial reference frame centered at the barycenter of the Solar System, defined by the measured positions of 212 extragalactic sources (mainly quasars). Although relativity implies that there is no true inertial frame, the extragalactic sources used to define the ICRF are so far away that any angular motion is essentially zero. The ICRF is now the standard reference frame used to define the positions of the planets (including the Earth) and other astronomical objects.

Motion of the Earth

- The **International Terrestrial Reference System** (**ITRS**) describes procedures for creating reference frames suitable for use with measurements on or near the Earth's surface. This is done in much the same way that a physical standard might be described as a set of procedures for creating a *realization* of that standard. The ITRS defines a geocentric system of coordinates using the SI system of measurement.
- An **International Terrestrial Reference Frame** (**ITRF**) is a realization of the ITRS. New ITRF solutions are produced every few years, using the latest mathematical and surveying techniques to attempt to realize the ITRS as precisely as possible. Due to experimental errorr, any given ITRF will differ very slightly from any other realization of the ITRF. Also, the difference between the latest WGS844 and the latest ITRF is only a few centimeters.

Relativity

1. Space-time is a 4-manifold with Minkowsky local structure
2. No prior geometry as part of Einstein’s principle of general covariance (MTW Gravitation)
3. Equations describing the laws of physics should be invariant under coordinate transformations
4. Coordinates are suitable, physically realizable, 1-forms on the space-time manifold

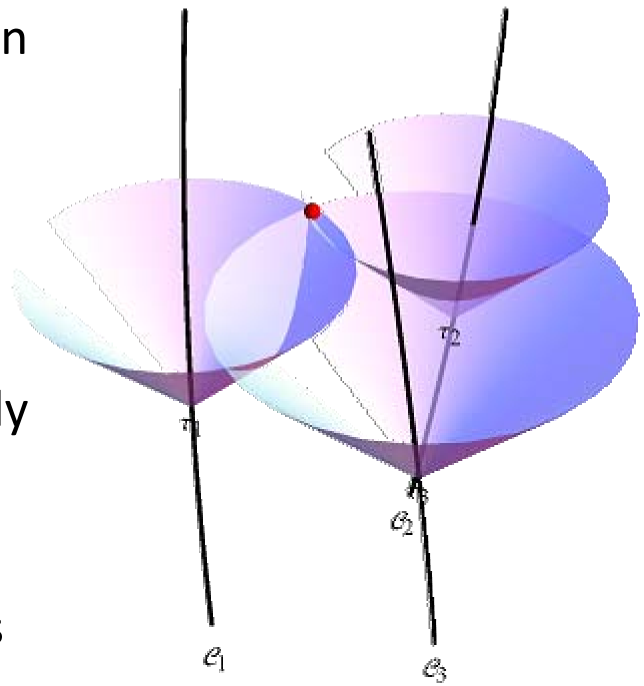
Mapping the space-time metric with a global navigation satellite system I

0-coordinates (emission coordinates) are suitable, physically realizable 1-forms generated by the same Hamiltonian as orbits of satellites generating them.

Schwarzschild space-time is a 4-manifold with Minkowsky local structure, but it appears as a prior geometry.

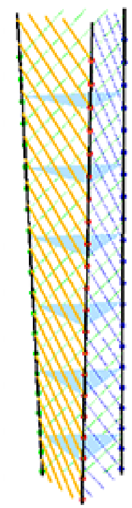
Lessons learnt:

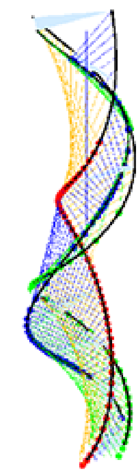
1. emission coordinates provide a useful system of 1-forms in the Schwarzschild space-time and can uniquely be transformed into standard Schwarzschild coordinates
2. The transformation from Schwarzschild to emission coordinates and vice versa has been numerically implemented and demonstrated to be stable and efficiently calculable also in comparison with classical post-Newtonian formalism
3. Non-gravitational perturbations entering the Schwarzschild -- emission coordinates transformation were shown to have a considerably weaker effect than timing uncertainty of present clocks .
4. Long term timing stability can be considerably improved by communicating emission coordinates between GNSS satellites.



Mapping the space-time metric with a global navigation satellite system **II**

- No prior geometry <u>but</u>: locally the geometry is Minkowskyan.
- A GNSS can detect deviations from Minkowsky geometry from data based solely on exchange of emission coordinates data.
- A perturbation approach to do so is suggested





Lessons learned:

- In order to find a place on Earth, to find a star or a planet in the sky, to understand laws of the Universe, one needs to understand dynamics. In other words, one needs to be able to write down a Lagrangian as a function of some suitable coordinates. In the framework of Newtonian physics, Cartesian coordinates are preferred, since the Hamiltonian, written with respect to these coordinates, shows the highest degree of symmetry, which leads to explicit constants of motion (without external forces).
- Tracks of free particle motion draw straight lines specified by 6 constants of motion, as required by Newton.
- In the case of relativistic GNSS, symmetry is equally important in the choice of coordinates. Local Minkowsky coordinates appear as a preferred choice, since such coordinatization must exist according to the founding axiom of general relativity.
- Emission coordinates provide natural means to propagate coordinates throughout space-time, since they are generated by the universal Lagrangian depending only on the curvature of space-time, which affects equations of motion of freely falling GNSS satellites in the same way .

Methods and tests:

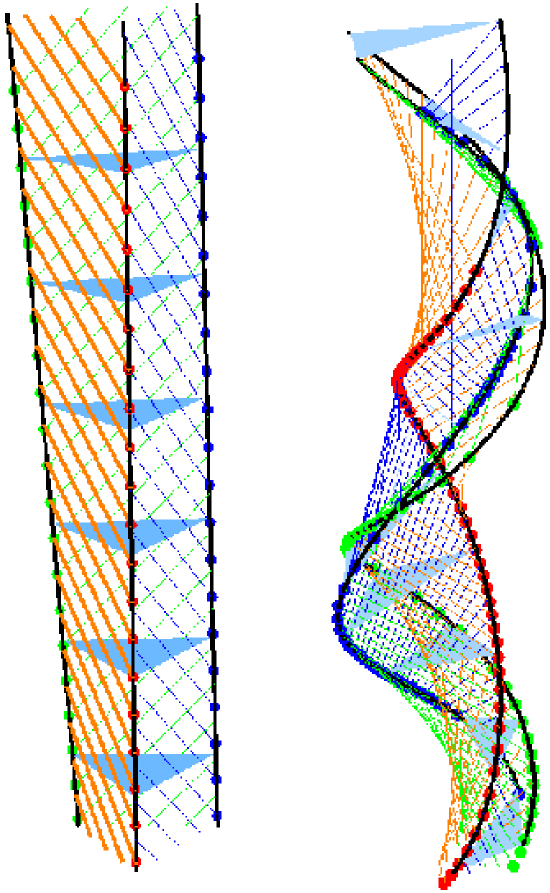
- The basic consequence of local Minkowsky structure of space-time is that in a small 3-volume covered by satellites of a GNSS constellation, the Lagrangian can be written in the local Minkowsky form:

$$L = \frac{1}{2}\eta_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} + \frac{1}{2}h^{(1)}{}_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} + \frac{1}{2}h^{(2)}{}_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} + \cdots$$

Where $\eta_{\mu\nu}$ is the Minkowsky metric tensor, and perturbations $h^{(1)}{}_{\mu\nu}$, $h^{(2)}{}_{\mu\nu}$ etc. decrease with their order. As a consequence, time-like and light-like solutions of equations of motion can be expressed in hamiltonian form as functions of constants of motion.

Mutual constants of motion

- Trajectories of two freely falling satellites and light rays exchanged between these two satellites form a two-manifold that can be described in terms of emission coordinates only. We have shown that emission coordinate data on this manifold allow the determination of some constants of motion of the two satellites involved. Furthermore, it was possible to demonstrate that four satellites exchanging data for a time T enclose a 4-volume of space-time and provide enough information to determine all constants of motion for the four satellites with respect to the assumed Lagrangian.



The ABC concept

(**A**utonomous **B**asis of **C**oordinates)

- An autonomous basis of coordinates is a basis, provided by a constellation of freely falling satellites around a central body (the Earth), which are allowed to communicate among themselves in order to determine mutual constants of all pairs.
- Mutual constants of motion, belonging to more than 4 satellites, are sufficient to establish motion with respect to the local Minkowski coordinates and thus provide the local Minkowsky frame of coordinates.

Properties of ABC

- The ABC framework provides a system of coordinates, in which dynamics is accurately expressed in the 4-volume covered by the constellation of satellites.
- ABC coordinates are defined with respect to the Lagrangian:

$$L = \frac{1}{2}\eta_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} + \frac{1}{2}{h^{(1)}}_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} + \frac{1}{2}{h^{(2)}}_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} + \cdots$$

Since $h^{(1)}$, $h^{(2)}$ … are progressively smaller quantities, their dynamic influence becomes apparent at later and later times. This can be understood in terms of classical perturbation theory, by which one can express solutions of a perturbed Hamiltonian with solutions of the unperturbed one with the former constants of motion becoming (slowly varying) functions of time. Thus, the highest perturbation order required to describe the growing 4-volume covered by the constellation of satellites grows with the length of existance of the constellation.

ABC coordinates can be related to ITRF and ICRF at any level of expansion. This relation is not axiomatic.

ABC in the Schwarzschild space-time

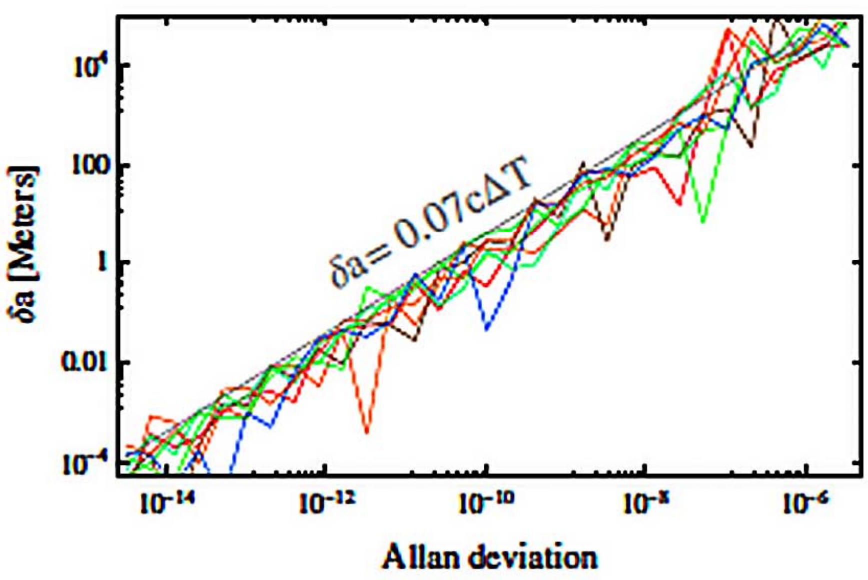
- Particular attention was paid to the Schwarzschild Lagrangian

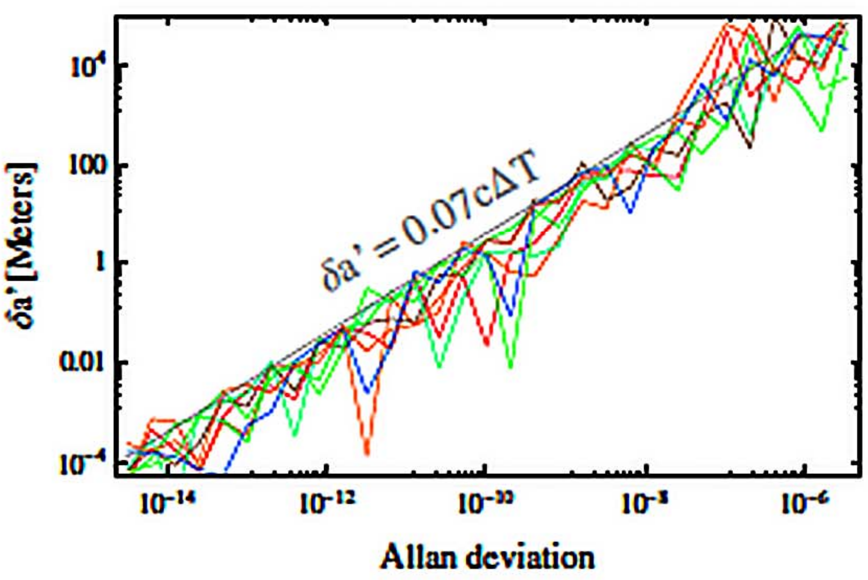
$$L_s = -\left(\frac{1-\frac{M^*}{2r}}{1+\frac{M^*}{2r}}\right)^2 c^2dt^2 + \left(1+\frac{M^*}{2r}\right)^2(dx^2+dy^2+dz^2),$$

which is a good approximation for the space-time metric in the vicinity of the Earth. Here $M^* = \frac{GM_{Earth}}{c^2}$ and $r = \sqrt{x^2+y^2+z^2}$. We calculated emission coordinates of satellites with respect to other satellites and numerically demonstrated that these data only are sufficient to reconstruct dynamics of satellites with respect to the original Schwarzschild frame via ABC technique, i.e. numerical simulations have shown that emission coordinate data can accurately be translated into constants of motion. We also demonstrated that a constellation consisting of more than 4 satellites is self consistent in the sense that frames defined by all possible quadruples are valid for all possible quadruples of satellites.

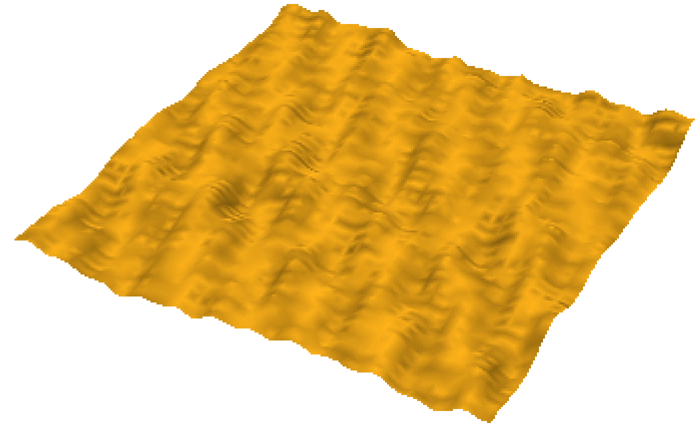
Accuracy of parameter determination from 4 orbits as limited by clock precision

b) mutual main axis errors

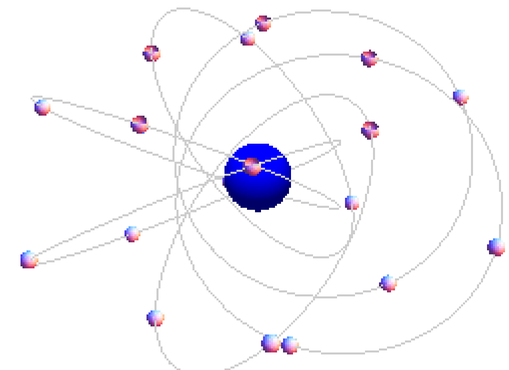




Why ABC should be added to ICRS and ITRS?



- growing accuracy of clocks
- very low noise communication between satellites in the outer space
- satellites can fly as drag free
- position accuracy of satellites may in the foreseeable future reach sub-millimeter or even sub-micron level.



- It is unlikely that the universe is flat from here to distant quasars at this level of precision
- Changing geometry of the Earth due to tides, volcanic activity, continental drift could be studied very accurately
- ABC would provide the most precise long scale timing standard